Optimization methods for measurement of returns to scale in the non-radial DEA models

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The non-radial DEA models\textsuperscript{1} possess some specific features. First, multiple reference sets may exist for a production unit. Second, multiple supporting hyperplanes may occur on optimal units of the frontier. Third, multiple projections (a projection set) may occur in the space of input and output variables. All these features cause certain difficulties under measurement of returns to scale of production units.

The non-radial DEA model can be written in the following form\textsuperscript{[1, 2]}

\begin{equation}
\begin{aligned}
\max h &= (C^{+T}S^{+} + C^{-T}S^{-}) \\
\text{subject to} & \sum_{j=1}^{n} X_{j}\lambda_{j} + S^{-} = X_{o}, \quad \sum_{j=1}^{n} Y_{j}\lambda_{j} - S^{+} = Y_{o}, \\
& \sum_{j=1}^{n} \lambda_{j} = 1, \quad S^{+} \geq 0, \quad S^{-} \geq 0, \quad \lambda_{j} \geq 0, \quad j = 1, \ldots, n, \\
\end{aligned}
\end{equation}

\textsuperscript{(1)}

here \(X_{j} = (x_{1j}, \ldots, x_{mj})\) and \(Y_{j} = (y_{1j}, \ldots, y_{rj})\) represent the observed inputs and outputs of production units \((X_{j}, Y_{j})\), \(j = 1, \ldots, n\), \(S^{-} = (s^{-1}, \ldots, s^{-m})\) and \(S^{+} = (s^{+1}, \ldots, s^{+r})\) are vectors of slack variables. The superscript “\(T\)” indicates a vector transpose. The components of the objective-function vectors \(C^{+}\) and \(C^{-}\) are specified as follows:

\begin{equation}
\begin{aligned}
c^{+}_{k} &= (m + r)^{-1} (\max\{x_{kj} | j = 1, \ldots, n\} - \min\{x_{kj} | j = 1, \ldots, n\})^{-1}, \\
c^{-}_{i} &= (m + r)^{-1} (\max\{y_{ij} | j = 1, \ldots, n\} - \min\{y_{ij} | j = 1, \ldots, n\})^{-1}, \\
& \quad k = 1, \ldots, m, \quad i = 1, \ldots, r.
\end{aligned}
\end{equation}

In the model\textsuperscript{(1)}, an efficiency score for unit \((X_{o}, Y_{o})\) is evaluated, where \((X_{o}, Y_{o})\) is any production unit from the set \((X_{j}, Y_{j})\), \(j = 1, \ldots, n\). If
the optimal value $h^*$ of the model is equal to zero, then unit $(X_o, Y_o)$ is considered efficient, if $h^* > 0$, then the unit is inefficient [1].

Banker et al. [1] proposed a two-stage approach to determine returns to scale in these models. Sueyoshi and Sekitani [2] showed that this approach may generate incorrect results in some cases. An interesting approach was proposed for measurement of returns to scale based on using strong complementary slackness conditions (SCSC) in the non-radial DEA models (SCSC/NM) [2]. The SCSC/NM model is written in the following form

$$\max \left\{ \eta \left| \begin{array}{c}
\theta X_o - \sum_{j=1}^{n} \lambda_j X_j \geq 0, \sum_{j=1}^{n} \lambda_j Y_j \geq Y_o, \sum_{j=1}^{n} \lambda_j = 1, \\
\lambda_j \geq 0, j = 1, \ldots, n, \\
v^T X_o = 1, -v^T X_j + u^T Y_j + u_0 \leq 0, j = 1, \ldots, n, \\
v \geq 0, u \geq 0, \\
\theta = u^T Y_o + u_0, \\
\lambda_j + v^T X_j - u^T Y_j - u_0 \geq \eta, j = 1, \ldots, n, \\
v - \sum_{j=1}^{n} \lambda_j X_j + \theta X_o \geq \eta, \\
u + \sum_{j=1}^{n} \lambda_j Y_j - Y_o \geq \eta, \eta \geq 0
\end{array} \right. \right\}. \quad (2)$$

The first six conditions are from the primal model (1), the next three conditions are from the dual problem, the tenth condition provides the equality of the objective functions of the primal and dual problems. The last three conditions express the SCSC constraints. In order to secure that strong complementarity is obtained the variable $\eta$ is entered as the objective function in (2).

Our theoretical consideration and computational experiments show that the SCSC/NM method may not be efficient from the computational point of view. Model SCSC/NM generates ill-conditioned basic matrices during the solution process, which results in “strange results” that do not coincide with the optimal solution of the corresponding non-radial DEA model. This naturally contradicts the optimization theory.

In our work we propose a two-stage approach to measure returns to scale in the non-radial DEA models. At first stage, an interior point,
belonging to the optimal face, is found using a special elaborated method. In our previous work [3] we proved that any interior point of a face has the same returns to scale as any other interior point of this face. At the second stage, we propose to determine the returns to scale at the interior point found in the first stage with the help of Banker and Thrall’s method [4] or using the direct method of Førsund et al. [5].

Our computational experiments documented that the proposed approach is reliable and efficient for solving real-life DEA problems.

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REFERENCES