Leontief’s model as a boundary value problem in optimal control

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1. Statement of problem. The optimal control problem with free right end on a fixed interval is considered in this paper. The dynamics of the process is described by a system of ordinary differential equations

\[ \frac{dx(t)}{dt} = D(t)x(t) + B(t)y(t), \quad t_0 \leq t \leq t_1, \quad x(t_0) = x_0 = 0, \] (1)

with the trajectories \( x(\cdot) \in PC^1([t_0, t_1], \mathbb{R}^n) \), and the controls \( y(\cdot) \in Y \).

\[ Y = \{ y(\cdot) \in PC([t_0, t_1], \mathbb{R}^n) \mid y_i(t) \in [y_i^-, y_i^+], i = 1, \ldots, n \}, \] (2)

\[ D(\cdot), B(\cdot) \in C([t_0, t_1], \mathbb{R}^{n \times n}). \] Let the right ends \( x_1 = x(t_1), y_1 = y(t_1) \) of the trajectories and controls satisfy the constraints

\[ x_1 = A_1 x_1 + y_1, \quad y_1 \geq 0, \] (3)

\( A_1 \in \mathbb{R}^{n \times n} \). Complementing the system (1) with a control \( y(\cdot) \in Y \) and solving it, we find the trajectory \( x(\cdot) \). When control changes within the set \( Y \) the right-hand ends \( x_1 \) of trajectories describe the attainable set, on which the following objective function is defined

\[ \varphi(x_1, y_1) = \varphi_1(x_1) + \varphi_2(y_1), \] (4)

where \( \varphi_1(x_1) \) and \( \varphi_2(y_1) \) are convex and differentiable in the variables \( x_1 \) and \( y_1 \), respectively.

We need to determine the optimal control \( y^*(\cdot) \in Y \) and the corresponding trajectory \( x^*(\cdot) \in PC^1([t_0, t_1], \mathbb{R}^n) \), subject to the system (1). At the same time their right ends \( y_1^* \) and \( x_1^* \) have to minimize the objective function (4) under constraints (2)–(3).

1 Here \( PC^1([t_0, t_1], \mathbb{R}^n) \) is the class of continuous vector-valued functions: \([t_0, t_1] \to \mathbb{R}^n\) with piecewise continuous derivatives; \( PC([t_0, t_1], \mathbb{R}^n) \) is the class of piecewise continuous vector-valued functions: \([t_0, t_1] \to \mathbb{R}^n\).
It is assumed that a solution exists, but not unique. Similar formulations of the problem, but without the terminal control, were examined in [1],[2]. In case \( \varphi(x_1, y_1) \equiv \varphi(y_1) \), the optimization problem is similar to the input-output economic model [3].

2. **Lagrangian, dual problem and boundary-value problem.**

Consider the Lagrangian for the optimization problem

\[
\mathcal{L}(p_1, \psi(\cdot); x(\cdot), y(\cdot)) = \varphi_1(x_1) + \varphi_2(y_1) +
\]

\[
+ \langle p_1, (I - A_1)x_1 - y_1 \rangle + \int_{t_0}^{t_1} \langle \psi(t), D(t)x(t) + B(t)y(t) - \frac{d}{dt} x(t) \rangle dt,
\]

defined for all \( x(\cdot) \in PC^1([t_0, t_1], \mathbb{R}^n), x_1 \in \mathbb{R}^n, y(\cdot) \in Y, y_1 \geq 0, p_1 \in \mathbb{R}^n, \psi(\cdot) \in PC^1([t_0, t_1], \mathbb{R}^n)' \) – the dual space.

The point \( (p_1^*, \psi^*(\cdot); x^*(\cdot), y^*(\cdot)) \) is called a saddle point of Lagrange function if for all \( x(\cdot) \in PC^1([t_0, t_1], \mathbb{R}^n), x_1 \in \mathbb{R}^n, y(\cdot) \in Y, y_1 \geq 0, p_1 \in \mathbb{R}^n, \psi(\cdot) \in PC^1([t_0, t_1], \mathbb{R}^n)' \) the following system holds

\[
\mathcal{L}(p_1, \psi(\cdot); x^*(\cdot), y^*(\cdot)) \leq \mathcal{L}(p_1^*, \psi^*(\cdot); x^*(\cdot), y^*(\cdot)) \leq \mathcal{L}(p_1^*, \psi^*(\cdot); x(\cdot), y(\cdot)).
\]

(5)

Here \( (x^*(\cdot), y^*(\cdot)) \) and \( (p_1^*, \psi^*(\cdot)) \) are called direct and dual variables.

According to the Kuhn-Tucker theorem, there exist \( p_1^* \) and \( \psi^*(\cdot) \), such that if the pair \( (x^*(\cdot), y^*(\cdot)) \) is a solution, then \( (p_1^*, \psi^*(\cdot); x^*(\cdot), y^*(\cdot)) \) is a saddle point of the Lagrangian. Converse is also true: if the pair \( (x^*(\cdot), y^*(\cdot)) \) satisfies the saddle point system (5), then it is a solution to the original problem.

Transforming the Lagrangian, as well as the saddle point system to conjugate form, and using the right-hand inequality of the system, we obtain the dual problem. Combining the direct and dual problems, we come to a boundary value problem

\[
\frac{d}{dt} x^*(t) = D(t)x^*(t) + B(t)y^*(t),
\]

\[
x^*(t_0) = x_0, y^*(\cdot) \in Y, t_0 \leq t \leq t_1,
\]

\[
x^*_1 = A_1 x^*_1 + y^*_1, \quad y^*_1 \geq 0,
\]

\[
\frac{d}{dt} \psi^*(t) + D^T(t)\psi^*(t) = 0, \quad \psi^*_1 = \nabla \varphi_1(x^*_1) + (I - A_1^T)p^*_1,
\]

\[
\varphi_2(y_1) - \varphi_2(y^*_1) - \langle y_1 - y^*_1, p^*_1 \rangle + \int_{t_0}^{t_1} \langle B^T(t)\psi^*(t), y(t) - y^*(t) \rangle dt \geq 0.
\]
3. Method of solution. The method of simple iteration is the simplest of the known numerical methods. However, in this case we are dealing with a saddle problem, for which the simple iteration method, generally speaking, not converge. Therefore, to solve the problem, we use an extra-proximal approach [4]–[5]:

1) prediction half-step

\[
\frac{d}{dt} x_k(t) = D(t)x_k(t) + B(t)y_k(t), \quad x_0 = x_0,
\]

\[
p_k^1 = p_k^1 + \alpha((I - A)x_k^1 - y_k^1),\]

\[
\frac{d}{dt} \psi_k(t) + D^T(t)\psi_k(t) = 0, \quad \psi_k^1 = \nabla \varphi_1(x_k^1) + (I - A_T^1)p_k^1,
\]

\[
(y_k^1, g^k(\cdot)) = \text{argmin} \left\{ \frac{1}{2}|y_1 - y_k^1|^2 + \alpha(\nabla \varphi_2(y_k^1) - p_k^1, y_1 - y_k^1) + \frac{1}{2} \int_{t_0}^{t_1} |y(t) - y_k(t)|^2 dt \right\};
\]

2) basic half-step

\[
\frac{d}{dt} x_k(t) = D(t)x_k(t) + B(t)y_k(t), \quad x_k(0) = x_0,
\]

\[
p_{k+1}^1 = p_k^1 + \alpha((I - A)x_k^1 - y_k^1),\]

\[
\frac{d}{dt} \psi_k(t) + D^T(t)\psi_k(t) = 0, \quad \psi_k^1 = \nabla \varphi_1(x_k^1) + (I - A_T^1)p_k^1,
\]

\[
(y_{k+1}^1, y_{k+1}^1(\cdot)) = \text{argmin} \left\{ \frac{1}{2}|y_1 - y_{k+1}^1|^2 + \alpha(\nabla \varphi_2(y_k^1) - p_k^1, y_1 - y_k^1) + \frac{1}{2} \int_{t_0}^{t_1} |y(t) - y_k(t)|^2 dt \right\};
\]

In the first half-step of each iteration we find the "forward" vectors \( p_k^1, \ \ y_k^1, \ \ g^k(\cdot) \) (those that should be at the next step in the simple iteration method). Then in the second half-step we use them to find the direction of future movement on the \((k + 1)\)-th iteration. From the current position in the \(k\)-th iteration, we take a step in that direction, and calculate \( p_{k+1}^1, \ \ y_{k+1}^1 \) and \( y_{k+1}^1(\cdot) \). Thus, when we choose the direction of movement, then
The following theorem on the convergence of the method was proved.

**Theorem 1.** Let the set of optimal trajectories \( x^*(\cdot) \) of the problem (1)–(4) be not empty and belong to the subspace \( PC^1([t_0,t_1],\mathbb{R}^n) \). If \( \varphi_1(x_1) \) and \( \varphi_2(y_1) \) are convex and differentiable functions with gradients satisfying a Lipschitz condition with constants \( L_1, L_2, D(t), B(t) \) are continuous matrices, \( Y \) is a set of the form (2). Then the sequence

\[
\{ |y^k_1 - y^*_1|_{\mathbb{R}^n}^2 + |p^k_1 - p^*_1|_{\mathbb{R}^n}^2 + \|y^k(\cdot) - y^*(\cdot)\|_{L_2}^2 \},
\]

generated by method (6) with the choice of parameter \( \alpha \) from the condition \( 0 < \alpha < \alpha_0 \), where \( \alpha_0 = \min(\frac{1}{\sqrt{2}L_2}, \frac{1}{\sqrt{2}L_2 + B_{\max}C_1}, \frac{1}{B_{\max}C_2}) \), \( B_{\max}, C_1, C_2 \) – some calculated constants, decreases monotonically. Moreover, every weakly converging subsequence of controls \( \{y^k(\cdot)\} \) converges weakly to the optimal control \( y^*(\cdot) \), and the corresponding subsequence of trajectories \( \{x^k(\cdot)\} \) converges to the optimal trajectory \( x^*(\cdot) \) in the uniform norm \( C^n[t_0,t_1] \).

If the sequence of controls \( \{y^k(\cdot)\} \) has at least one strong limit as \( k \to +\infty \) then the process \( \{x^k_1, y^k_1, p^k_1; x^k(\cdot), y^k(\cdot), \psi^k(\cdot)\} \) converges to a solution of the problem \( x^*_1, y^*_1, p^*_1; x^*(\cdot), y^*(\cdot), \psi^*(\cdot) \), with respect to the variables \( y(\cdot), y_1, p_1 \) – monotonically.

**4. Conclusions.** The optimal control problem with free right end and linear differential equations constraints is considered. The right-hand ends of controls and trajectories generate a finite-dimensional Cartesian product, on which a minimum of the objective function is defined under constraints such as input-output model of Leontief.

To solve this problem we suggest an iterative method of extra-proximal type, consisting in the construction of the functional sequences of trajectories and controls. We have proved that the sequences of controls, trajectories, conjugate trajectories, as well as the sequences in finite-dimensional spaces of primal and dual variables converge monotonically in norm to solution of the problem.

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REFERENCES


