Existence theorems for elliptic equations in unbounded domains

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We consider the first boundary value problem for elliptic systems defined in unbounded domains, which solutions satisfy the condition of finiteness of the Dirichlet integral also called the energy integral

\[ \int_{\Omega} |\nabla u|^2 dx < \infty. \]

Basic concepts

Let \( \Omega \) is an arbitrary open set in \( \mathbb{R}^n \). As is usual, by \( W_{2, \text{loc}}^1(\Omega) \) we denote the space of functions which are locally Sobolev, i.e.

\[ W_{2, \text{loc}}^1(\Omega) = \{ f : f \in W_2^1(\Omega \cap B_{\rho}^x), \forall \rho > 0, \forall x \in \mathbb{R}^n \}, \]

where \( B_{\rho}^x \) – open ball with center at point \( x \) and with radius \( \rho \). If \( x = 0 \) then we will write \( B_{\rho} \). We will denote by \( \tilde{W}_{2, \text{loc}}^1(\Omega) \) set of functions from \( W_{2, \text{loc}}^1(\mathbb{R}^n) \), which is the closure of \( C_0^\infty(\Omega) \) in the system of seminorms \( \| u \|_{W_2^1(K)} \), where \( K \subset \mathbb{R}^n \) are various compacts. Let denote by \( L_2^1(\Omega) \) a space of generalized functions in \( \Omega \), which first derivatives belong to \( L_2(\Omega) \) [4], in other words

\[ L_2^1(\Omega) = \{ f \in \mathcal{D}'(\Omega) : \int_{\Omega} |\nabla f|^2 dx < \infty \}. \]

Let \( \omega \subseteq \mathbb{R}^n \) is an open set, \( K \subset \omega \) is a compact. We will denote by \( \Phi_{\varphi}(K, \omega) \) the set of functions \( \psi \in C_0^\infty(\omega) \) such that \( \psi = \varphi \) in the neighborhood of \( K \), or in other words \( \psi - \varphi \in W_{2, \text{loc}}^1(\mathbb{R}^n \setminus K) \).

Let’s define a capacitance of a compact \( K \) relative to the set \( \omega \) [4]:

\[ \text{cap}_{\varphi}(K, \omega) = \inf_{\psi \in \Phi_{\varphi}(K, \omega)} \int_{\omega} |\nabla \psi|^2 dx. \]
The capacitance of arbitrary closed set $E \subset \omega$ in $\mathbb{R}^n$ is defined by the formula $\text{cap}_\varphi(E, \omega) = \sup_{K \subset E} \text{cap}_\varphi(K, \omega)$. If $\omega = \mathbb{R}^n$, then instead of $\text{cap}_\varphi(E, \mathbb{R}^n)$ we will write $\text{cap}_\varphi(E)$.

**Problem statement**

Let $L$ is a divergent operator

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right),$$

where $a_{ij}$ are bounded measurable functions in $\mathbb{R}^n$ satisfying condition

$$\gamma |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j, \quad \xi \in \mathbb{R}^n, \gamma > 0.$$

The solution of the Dirichlet problem

$$\begin{align*}
Lu & = 0 \quad \text{in } \Omega \\
{u}|_{\partial\Omega} & = \varphi,
\end{align*}$$

where $\varphi \in W^{1,2}_{2,loc}(\mathbb{R}^n)$, is a function $u \in W^{1,2}_{2,loc}(\Omega)$ such that:

1) $u - \varphi \in \tilde{W}^{1,2}_{2,loc}(\Omega)$, i.e. $(u - \varphi)\mu \in \tilde{W}^{1,2}(\Omega)$ for any function $\mu \in C^\infty_0(\mathbb{R}^n)$;

2) function $u$ has bounded Dirichlet integral

$$\int_{\Omega} |\nabla u|^2 dx < \infty;$$

3) $\int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx = 0$

for any function $\varphi \in C^\infty_0(\Omega)$. 


Basic results

Theorem 1. Let’s cap \(\varphi - c(R^n \setminus \Omega) < \infty\) for some constant \(c \in \mathbb{R}^n\). Then the problem (1) has a solution.

Theorem 2. Let the problem (1) has a solution and it is true that
\[
\int_{\mathbb{R}^n \setminus \Omega} |\nabla \varphi|^2 dx < \infty.
\]
Then there is such constant \(c \in \mathbb{R}^n\), that cap \(\varphi - c(R^n \setminus \Omega) < \infty\).

Theorem 3. Let \(n \geq 3\). Then cap \(\varphi - c(R^n \setminus \Omega) < \infty\) if and only if
\[
\sum_{k=N}^{\infty} \text{cap}_{\varphi - c}((B_{2k+1} \setminus B_{2k-1}) \cap (\mathbb{R}^n \setminus \Omega), B_{2k+2} \setminus B_{2k-2}) < \infty
\]
for some \(N \in \mathbb{N}\).

Particular cases

Let consider the space \(\mathbb{R}^n\) with a set of coordinates \((x_1, x_2, \ldots, x_n)\) and let \(\varphi_\alpha = (1 + |x_1|)^\alpha\). Domain \(\Omega_{1,i}\) is upper half-plane relative to \(x_i\), where \(i \neq 1\), in other words \(\Omega_{1,i} = \{(x_1, x_2, \ldots, x_n) | x_i \geq 0, i \neq 1\}\). Domain \(\Omega_2\) is the outer part of the space formed by surface of revolution relative to \(x_1\) of the curve from Fig. 1.

![Fig. 1: Domain \(\Omega_2\)](image)

\[x_2 = |x_1|^\beta, \ \beta < 0\]
Corollary 1. Let $n \geq 2$. Then for the domain $\Omega_{1,i}$ and for bounded function $\varphi_{\alpha}$ the existence of solutions of the problem (1) is equivalent to either an inequality $\alpha < -\frac{1}{2}$ or $\alpha = 0$.

Corollary 2. Let $n \geq 3$. Then for the domain $\Omega_{2}$ and for bounded function $\varphi_{\alpha}$ the existence of solutions of the problem (1) is equivalent to either an inequality $\alpha < -\frac{1 + \beta(n - 3)}{2}$ or $\alpha = 0$.

REFERENCES