KLEINIAN GROUPS AND HOLOMORPHIC DYNAMICS

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It is known as a correspondence between iteration of rational maps and Kleinian groups, and
is usually designated as the Sullivan’s dictionary. This dictionary enumerates analogies between
iteration of holomorphic endomorphisms and action of Kleinian groups. We propose to study
explicitly examples establishing the correspondence between the two theories: iteration theory
and Kleinian groups.

We present, in Sec. 2, part of the Sullivan’s dictionary. We also present some known results about
Kleinian groups which allow us to give explicit examples of relations between iteration theory and
Kleinian groups.

In this work we explore some relations between iteration theory and theory of Kleinian groups. There
is already work done in that direction. In the case of Fuchsian groups there is the result of Bowen and
Series [1979] which guarantees the existence of a boundary map, orbit equivalent to the action of the
Fuchsian group on the hyperbolic plane. We use this result in Sec. 3. Brooks and Matelski [1981] used
Kleinian groups and complex iteration in relation with the question of deciding when a given two-
generator group is discrete, using the Jorgensen inequality. This approach is also followed by Gehring
and Martin [1989].

On the other hand, the existence is known of many analogies between iteration theory of holono-
orphic transformations and the theory of Kleinian groups. These analogies are collected in what is
usually called Sullivan’s dictionary. The number of these analogies and the importance of the results
suggest deep connections between them.

We present, in Sec. 2, part of the Sullivan’s dictionary. We also present some known results about
Kleinian groups which allow us to give explicit examples of relations between iteration theory and
Kleinian groups.

In Sec. 3 we consider a particular kind of Kleinian groups, the Fuchsian groups, and using
the result in [Bowen & Series, 1979], we give explicit relations between these groups and the iteration
of piecewise fractional linear transformations. We build subshifts associated to Hecke groups, a
one-parameter family of Fuchsian groups. In Theorem 3.3 to every Hecke group $H_q$, we associate finite
type subshift defined by a transition matrix with dimension $q-2$. In Theorem 3.4 and Corollary 3.5 we
characterize the topological entropy of the subshifts associated to the Hecke groups. The methods here
introduced can be generalized to other families of Fuchsian groups.

Finally in Sec. 4 we study automorphisms of Kleinian groups and we use it to establish a dif-
ferent kind of relation between iteration theory and Kleinian groups. We focus on a one-parameter
holomorphic family of Kleinian groups which
corresponds to the family of quasi-Fuchsian groups introduced in [Wright, 1988]. See also [Keen et al., 1993]. Using iteration theory we identify where these groups are in the parameter space. In Theorem 4.2, we state that the parameters for those Kleinian groups do not belong to the Cartesian product of the Mandelbrot set and the filled Julia set of a quadratic map. Thus, a sequence of subgroups defined by iterating the commutator of the generators, have their traces given by a polynomial transformation which implies discreteness criteria for subgroups in $PSL_2(\mathbb{C})$.

1. Preliminaries

We can define Kleinian Group as a discrete subgroup $\Gamma$ of $PSL_2(\mathbb{C})$. An element of $PSL_2(\mathbb{C})$ can be represented by a matrix of $SL_2(\mathbb{C})$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - bc = 1$, reminding the fact that $PSL_2(\mathbb{C}) \simeq SL_2(\mathbb{C})/\{\pm I\}$. The elements of $PSL_2(\mathbb{C})$ act on $\mathbb{H}^3$ as orientation-preserving isometries, and act on $\mathbb{C}_\infty$ as automorphisms, $\gamma(z) = (az+b)/(cz+d)$, Möbius transformations that preserves orientation.

The set $\Gamma(p) = \{\gamma(p) : \gamma \in \Gamma\}$ is denoted as the orbit of $p \in \mathbb{H}^3$. The set $\Gamma(p)$ has accumulation points only on the Riemann Sphere $\mathbb{C}_\infty = \partial \mathbb{H}^3$. These points form a closed set called the limit set of $\Gamma$, $\Lambda(\Gamma)$ (see Fig. 1, for $\Gamma_a = \langle F, G_a \rangle$

$$= \left\langle \begin{pmatrix} 1 & 0 \\ -I & 1 \end{pmatrix}, \begin{pmatrix} a & \sqrt{a^2-1} \\ \sqrt{a^2-1} & a \end{pmatrix} \right\rangle,$$

with $a = 2$). The set $\Omega(\Gamma) = \mathbb{C}_\infty \setminus \Lambda(\Gamma)$ is called the regular set.

According to the type of limit set $\Lambda(\Gamma)$, a Kleinian group is called:

Elementary, if $\Lambda(\Gamma)$ consists of at most two points. Otherwise it is nonelementary.

Fuchsian, if $\Gamma$ has an invariant disc $D$.

The elements in $PSL_2(\mathbb{C})$ can be classified according to their set of fixed points or, which is equivalent, by the values of the square of the traces for the corresponding matricial representatives:

$g \in PSL_2(\mathbb{C})$ is parabolic (resp. elliptic, hyperbolic, loxodromic) if $Tr^2(g)$ is 4 (resp. $\in [0, 4[, \in [4, +\infty], \notin [0, +\infty]$).

Now recall some basic notions on dynamics of rational maps.

Given a rational application $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ we define:

The set of escaping points for $f$

$$\mathcal{E}(f) = \{ z \in \mathbb{C} : \lim_{n \to +\infty} f^n(z) = \infty \}$$

The filled Julia set $\mathcal{K}(f) = \mathbb{C} - \mathcal{E}(f)$. 

Fig. 1. Limit set of the Kleinian group $\Gamma_a$ with $a = 2$.

Fig. 2. Julia set of $g_\beta(z) = z(1 + \beta - z)^2$, with $\beta = 2i$. 

Fig. 2. Julia set of $g_\beta(z) = z(1 + \beta - z)^2$, with $\beta = 2i$. 


The boundary $\mathcal{J}(f) = \partial \mathcal{K}(f)$ is called the Julia set of $f$ (see Fig. 2).

The set $\mathcal{M}(f) = \{ c \in \mathbb{C}^n; \text{the critical points of } f \text{ having a bounded orbit under } f \}$ is called the Mandelbrot set.

2. Sullivan’s Dictionary and Some Known Facts about Kleinian Groups

The Sullivan’s dictionary consists of analogies between iteration of rational maps and action of Kleinian groups. A more complete description of this dictionary can be found in [Sullivan, 1985]. We present here part of it.

<table>
<thead>
<tr>
<th>Complex Dynamics</th>
<th>Kleinian Groups</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f : \mathbb{C}<em>\infty \to \mathbb{C}</em>\infty$</td>
<td>$\Gamma$ Kleinian group, non-elementary and finitely generated,</td>
</tr>
<tr>
<td>rational map</td>
<td>Limit set $\Lambda(\Gamma)$,</td>
</tr>
<tr>
<td>of degree $&gt; 2,$</td>
<td>Regular set $\Omega(\Gamma)$,</td>
</tr>
<tr>
<td>Julia set $\mathcal{J}(f),$</td>
<td>Parameter space $\text{Par}(\Gamma)$,</td>
</tr>
<tr>
<td>Fatou set $\mathcal{F}(f),$</td>
<td>$\Lambda(\Gamma) : \text{smallest}$ closed nonempty invariant set,</td>
</tr>
<tr>
<td>Mandelbrot set $\mathcal{M}(f),$</td>
<td>$# \mathcal{J}(f) \geq 3$,</td>
</tr>
<tr>
<td>$\mathcal{J}(f) : \text{smallest}$ closed nonempty invariant set,</td>
<td>repelling periodic points are dense in $\mathcal{J}(f)$,</td>
</tr>
<tr>
<td>$# { \text{components of } \mathcal{F}(f) }$</td>
<td>$# { \text{components of } \Omega(\Gamma) }$</td>
</tr>
<tr>
<td>$= 0, 1, 2,$</td>
<td>$= 0, 1, 2,$</td>
</tr>
<tr>
<td>or $\infty$,</td>
<td>or $\infty$,</td>
</tr>
</tbody>
</table>

No wandering domains theorem:
Every $f$ rational with a finite number of singular values has no wandering domain, degree $k$ gives $2k - 2$ complex parameters.

In what follows we present some results that are relevant in the theory of Kleinian groups and which are useful for the objectives of this work. They can be found, for example, in [Beardon, 1983].

Concerning the question of deciding if a given group of Möbius transformations is discrete there is an important result on groups with two generators.

**Theorem 2.1.** (Jorgensen inequality) *If the group generated by $A, B \in \text{PSL}_2(\mathbb{C}), \langle A, B \rangle,$ is a non-elementary Kleinian group then*

$$|\text{Tr}^2(B) - 4| + |\text{Tr}(\langle A, B \rangle) - 2| \geq 1.$$  

This gives a necessary condition for a given group to be a discrete nonelementary group. As far as we know, there is not a necessary and sufficient condition for a given group to be Kleinian. Nevertheless there is a result that tells us that it is enough to decide the question to groups with two generators.

**Theorem 2.2.** A nonelementary subgroup of $\text{PSL}_2(\mathbb{C}), \Gamma,$ is discrete if and only if each two-generator subgroup of $\Gamma$ is discrete.

**Remark 2.3.** There is a recent result by Wang and Yang [2000] which establishes that a nonelementary subgroup of $\text{PSL}_2(\mathbb{C}), \Gamma,$ is discrete if and only if each subgroup of $\Gamma$ generated by two loxodromic elements is discrete.

Concerning the two-generator groups, it is known that a two-generator group $\Gamma = \langle A, B \rangle$ is determined, up to conjugacy by numbers $\alpha = \text{Tr}^2(A) - 4, \beta = \text{Tr}^2(B) - 4$ and $\gamma = \text{Tr}(\langle A, B \rangle) - 2$ provided that $\gamma \neq 0$ (see [Gehring et al., 1997]). These values are called the parameters of the group $\Gamma$.

We have then
$$\text{par}(\Gamma) = (\text{Tr}^2(A) - 4, \text{Tr}^2(B) - 4, \text{Tr}[A, B] - 2) = (\alpha, \beta, \gamma) \in \mathbb{C}^3.$$  

It would be interesting to describe the subset of $\mathbb{C}^3$ corresponding to the parameters of Kleinian two-generator groups. There is some work done in that direction, see [Gehring et al., 1997; Baribeau & Ransford, 2000].

A result that is very useful when trying to prove if a given group is a nonelementary Kleinian group, and which is a step forward in the direction of iteration theory is

**Proposition 2.4.** For $A \in \text{PSL}_2(\mathbb{C})$ and
$$\theta_A : \text{PSL}_2(\mathbb{C}) \to \text{PSL}_2(\mathbb{C})$$
$$\theta_A(B) = BAB^{-1}$$
if a Kleinian group $\langle A, B \rangle$ satisfies $\theta^n_A(B) = A$ for some integer $n$, and $A$ is not of order 2 then $\text{Fix}(A)$ is invariant under $B$. In particular $\langle A, B \rangle$ is elementary.
3. Relations Between Iteration Theory and Fuchsian Groups

3.1. Hecke groups

In this section we give an explicit relation between iteration theory and a particular case of Kleinian groups — Fuchsian groups. The possibility of associating to a Fuchsian group a map is known, defined on the boundary of \( \mathbb{H}^2 \), \( \mathbb{R} \cup \{\infty\} \), that is orbit equivalent to the group [Bowen & Series, 1979]. We build explicitly this map for the Hecke groups.

A Hecke group, \( H_q \), is a one-parameter family of Fuchsian groups generated by the following transformations

\[
A(z) = \frac{1}{u(1-z)} \quad \text{and} \quad B(z) = z - 1.
\]

If \( u < 4 \), the groups are discrete only for the parameter of the form \( u = 4 \cos^2(\pi/q) \), \( q \) integer \( q \geq 3 \) (the case \( q = 3 \) corresponds to the modular group \( \text{PSL}_2(\mathbb{Z}) \)).

The boundary map is built taking into account an appropriate fundamental domain for the group. There are many possible choices. We have chosen as the fundamental domain for the Hecke groups the polygon delimited by the vertices \( i, i+1, \left(1+i\sqrt{3}\right)/2, \infty \). In fact we build a boundary map for each group \( H_q \), so we have a one-parameter family of maps, \( f_q \).

The family of piecewise fractional-linear transformations associated to the Hecke groups, are given by

\[
f_q(x) = \begin{cases} 
B^{-1}(x) & \text{if } x \leq -1 \\
A^{-1}(x) & \text{if } -1 < x \leq 0 \\
A(x) & \text{if } 0 < x \leq 1 \\
B(x) & \text{if } x \geq 1 
\end{cases}
\]

For each map, \( f_q \), we can build a Markov partition and the associated Markov matrix, \( M_q \).

**Example 3.1.** For the case \( q = 3 \) we have the following partition

\[
I_1 = ]-\infty, -1[, \quad I_2 = ]-1, 0[, \quad I_3 = ]0, 1[ \\
I_{41} = ]1, 2[, \quad I_{42} = ]2, +\infty[
\]

and the matrix

\[
M_3 = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}
\]

**Example 3.2.** We present here the matrix for the case \( q = 4 \)

\[
M_4 = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

In general we have the following result

**Theorem 3.3.** To every Hecke group \( H_q \) we can associate the Markov matrix

\[
M_q = \begin{pmatrix}
1 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 1 & 1 & 1 & 1
\end{pmatrix}
\]

with \( \dim(M_q) = q + 2 \).
Proof. The partition for the general case is given by the following

\[
I_1 = [-\infty, -1] \\
I_2 = [-1, 0] \\
I_{31} = [0, x_1] \\
I_{32} = [x_2, x_3] \\
\ldots \\
I_{3(q-2)} = [x_{q-3}, 1] \\
I_{41} = [1, 1 + x_1] \\
I_{42} = [1 + x_1, +\infty]
\]

with

\[
I_1 \cup I_2 \bigcup_{i=2}^{q-2} I_{3i} \cup I_{41} \cup I_{42} = \mathbb{R} \setminus \{\infty\}
\]

and where \(x_1 = 1/u\), and \(x_i = A(x_{i-1})\), \(A\) being one of the group generators, defined above. Looking for the orbits of the critical points we can determine the possible transitions by the action of the boundary map, \(f_q\). We have

\[
f_q(I_1) = I_1 \cup I_2 \\
f_q(I_2) = I_{42} \\
\ldots \\
f_q(I_{3i}) = I_{3(i+1)}, \quad i = 1, \ldots, q-3 \\
\ldots \\
f_q(I_{3(q-2)}) = I_{41} \cup I_{42} \\
f_q(I_{41}) = I_{31} \\
f_q(I_{42}) = \bigcup_{i=2}^{q-2} I_{3i} \cup I_{41} \cup I_{42}
\]

So the matrix, that codifies these transitions is \(M_q\) presented in the statement. \(\square\)

With matrices, it is possible to determine the topological entropy that becomes an invariant for the Hecke group via topological dynamics. The topological entropy may be determined as the logarithm of the growth number, \(s(f_q)\),

\[
s = \lim_{n \to \infty} \frac{n}{\ell(f_q^n)}
\]

where \(\ell(f_q^n)\) is the lap number of \(f_q^n\). It is known to be equal to the spectral radius of the matrix \(M_q\) (see [Lind & Marcus, 1995]).

We present some of the growth numbers \(s\) for the groups associated with the first values of \(q\):

<table>
<thead>
<tr>
<th>(q)</th>
<th>(s) (aprox.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.61803</td>
</tr>
<tr>
<td>4</td>
<td>1.83929</td>
</tr>
<tr>
<td>5</td>
<td>1.92756</td>
</tr>
<tr>
<td>6</td>
<td>1.96595</td>
</tr>
<tr>
<td>7</td>
<td>1.98358</td>
</tr>
</tbody>
</table>

It is clear that the growth numbers approach the value 2. In fact we prove the following results

**Theorem 3.4.** The topological entropy of the subshift associated to the Hecke groups \(H_q\) is given by \(h_t = \log s\), where \(s\) (spectral radius of the matrix \(M_q\)) is the greatest real zero of the polynomial \(p_q(t) = t^{q-1} - t^{q-2} - \cdots - t - 1\).

Proof. Consider the matrix \(M_q - tI_{q+2}\) where \(I_{q+2}\) is the \((q+2) \times (q+2)\) identity matrix. This matrix is reducible and is decomposed in the following way

\[
M_q - tI_{q+2} = \begin{pmatrix}
1 - t & 1 & * & \cdots & * \\
0 & -t & * & \cdots & * \\
0 & 0 & b_{11} - t & \cdots & b_{1,q} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & b_{q,1} & \cdots & b_{q,q} - t
\end{pmatrix}
\]

so \(\det(M_q - tI_{q+2}) = t(t-1)\det(B_q - tI_q)\). If we set \(N_q - tI_q = P(B_q - tI_q)Q\), where \(P, Q\) are convenient elementary matrices, we obtain a matrix \(N_q\)

\[
N_q - tI_q = \begin{pmatrix}
C_{q-1} - tI_{q-1} & 0 \\
* & -t
\end{pmatrix}
\]

such that \(\det(N_q - tI_q) = \det(B_q - tI_q)\) and \(C_{q-1}\) is the companion matrix with characteristic polynomial \(t^{q-1} - t^{q-2} - \cdots - t - 1\). \(\square\)

**Corollary 3.5.** The entropies \(h_t\), associated to the subshifts \(M_q\), for the Hecke groups \(H_q\), are bounded above by \(\log 2\).

Proof. Relating the polynomial \(p_q(t) = t^{q-1} - t^{q-2} - \cdots - t - 1\), as \(q \to \infty\), with the geometric series, it is easy to see that the zero of \(p_\infty(t)\) is \(t = 2\). \(\square\)

It is relevant to note that like in the Kneading Theory [Milnor & Thurston, 1988] the dynamics here is completely determined by the symbolic sequences (kneading sequences) corresponding to the
the parameters, the partition is always the same, and so the topological information is the same. The surface obtained by $\mathbb{H}^2/G$ is the independent of the parameters values.

The generators are

$$A(z) = \lambda^2 z, \quad \text{and} \quad B(z) = \frac{pz + p^2 - z^2}{z + p}$$

with

$$r = p \frac{\lambda^2 - 1}{\lambda^2 + 1}, \quad p, \quad \lambda \in \mathbb{R}.$$ 

The family of associated boundary maps is the following

$$f_{\lambda,p}(x) = \begin{cases}
A^{-1}(x) & \text{if } x \leq -p - r \\
B^{-1}(x) & \text{if } -p - r < x \leq -p + r \\
A(x) & \text{if } -p + r < x \leq p - r \\
B(x) & \text{if } p - r < x \leq p + r \\
A^{-1}(x) & \text{if } p + r < x
\end{cases}$$

The partition and consequently the Markov matrix

$$M = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}$$

are independent of the parameters $\lambda, p$.

The topological entropy associated with the Markov subshift is log 3, which indicates the fact that the group is free. See in Fig. 4 a fundamental domain and Fig. 5 the map $f_{\sqrt{2},1}$.

![Fig. 4. Fundamental domain for $G_{\lambda,p} = \langle A_{\lambda,p}, B_{\lambda,p} \rangle$, $\lambda = \sqrt{2}$ and $p = 1$.](image-url)
4. Some Special Automorphisms of the Kleinian Groups

As was said before, given a group generated by two transformations, it is possible to assign parameters that characterize the group up to conjugacies (see [Gehring et al., 1997]). We write

$$\text{par}(\Gamma) = (\text{Tr}^2(A) - 4, \text{Tr}^2(B) - 4, \text{Tr}[A, B] - 2)$$

$$= (\alpha, \beta, \gamma) \in \mathbb{C}^3$$

With this notation the Jorgensen inequality is expressed by $|\beta| + |\gamma| \geq 1$. We have that if $|\beta| + |\gamma| < 1$ then the group $\Gamma$ is either not discrete or elementary.

It is possible to explore some of the two-generator subgroups of $\Gamma$ using automorphisms in the group. If we consider the following automorphism

$$\theta_B : \text{PSL}_2(\mathbb{C}) \rightarrow \text{PSL}_2(\mathbb{C})$$
$$\theta_B(A) = ABA^{-1}$$

it is possible to prove that, defining

$$z_n = \text{Tr}[B, \theta_B^n A] - 2,$$

we have

$$z_n = z_{n-1}(z_{n-1} - \beta).$$

If we iterate $f_\beta(z) = z(z - \beta)$, with the initial condition $z_0 = \gamma$, we are obtaining the subgroups $\Gamma_n$ with

$$\text{par}(\Gamma_n) = (\beta, \beta, z_n) = (\beta, \beta, z_{n-1}(z_{n-1} - \beta)).$$

Conclusions on discreteness of the group $\Gamma$ can be taken from this iteration since if we know that some subgroup is not discrete then the group itself cannot be discrete. In [Brooks & Matelski, 1981] and [Gehring & Martin, 1989] the authors identify the Jorgensen inequality as the basin attraction of the fixed point.

We consider the set $(\mathcal{M}(f_\beta) \times \mathcal{K}(f_\beta))^\circ$, the interior of the Cartesian product of the Mandelbrot set, $\mathcal{M}(f_\beta)$ and the filled Julia set $\mathcal{K}(f_\beta)$ for the map $f_\beta(z)$. Elements $(\beta, \gamma) \in (\mathcal{M}(f_\beta) \times \mathcal{K}(f_\beta))^\circ$ correspond to nondiscrete or elementary groups $\Gamma$ with $\text{par}(\Gamma) = (\alpha, \beta, \gamma)$.

With the automorphism $\theta_B$ only a particular path in the group is explored. It is possible to give different kinds of automorphisms with different associated polynomials. Iteration with group automorphisms corresponds to iteration of polynomials with the parameter $\gamma$ as initial condition. Another example is

$$\phi_B : \text{PSL}_2(\mathbb{C}) \rightarrow \text{PSL}_2(\mathbb{C})$$
$$\phi_B(A) = ABA^{-1}BA$$

The polynomial associated to $\phi_B$ is $g_\beta(z) = z(1 + \beta - z)^2$, this time is a cubic (see Fig. 2 for an example of Julia set). The sequence of subgroups is now the following $\text{par}(\Gamma_n) = (\alpha_n, \beta, z_{n-1}(1 + \beta - z_{n-1})^2)$. For the automorphism $\phi_B$ we have the set $(\mathcal{M}(g_\beta) \times \mathcal{K}(g_\beta))^\circ$ associated to the map $g_\beta(z)$. In general we have the following relations

$$\psi_B \rightarrow p_\beta \leftrightarrow (\mathcal{M}(p_\beta) \times \mathcal{K}(p_\beta))^\circ$$

where $\psi_B$ is an group automorphism, $p_\beta$ is the associated polynomial, and $\mathcal{M}(p_\beta), \mathcal{K}(p_\beta)$ the corresponding Mandelbrot and filled Julia set.

In what follows, we consider the one-parameter family of Kleinian groups $G_\xi = \langle A, B_\xi \rangle$, with $\xi \in \mathbb{C}$, where the generator representatives in $SL_2(\mathbb{C})$ are given by

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_\xi = \begin{pmatrix} -i\xi & -i \\ -i & 0 \end{pmatrix}$$

Note that $\text{Tr}[A, B] = -2$. We are going to identify where these groups are in the parameter space, i.e. we are looking for $\text{par}(G_\xi)$.

Such groups were introduced in [Wright, 1988] and [Keen et al., 1993] where the authors identify the cusp groups. For each $\xi \in \mathbb{C}$ such $\text{Tr}(W_{p/q}) = \pm 2$, $G_\xi$ corresponds to a cusp group. The words $W_{p/q}$ are built recursively by the following process

$$W_{1/0} = A \quad W_{-1/1} = A^{-1}B_\xi$$
$$W_{1/1} = AB_\xi \quad W_{0/1} = B_\xi$$
$$W_{(p+r)/(q+s)} = W_{p/q}W_{r/s}$$

Fig. 5. Graph of the map $f_{\sqrt{\xi}, 1}$. 

Kleinian Groups and Holomorphic Dynamics 1965
with \[
\frac{p}{q} = 1 + r/s > r/s.
\]

The traces of this group words are polynomials in the variable \(\xi \in \mathbb{C}\). The recursion on the words, and the trace identity

\[
\text{Tr}(XY) = \text{Tr}(X)\text{Tr}(Y) - \text{Tr}(XY^{-1})
\]

for any matrices \(X, Y \in SL_2(\mathbb{C})\) allow a definition of an iterative procedure to explore paths in the group to determine for which \(\xi\) corresponds to a cusp group. Let \(c = \text{Tr}(Y)\), \(t_0 = \text{Tr}(X)\) and \(t_1 = \text{Tr}(XY)\), we have

\[
t_{n+2} = ct_{n+1} - t_n
\]

where \(t_n = \text{Tr}(XY^n)\), \(X = W_{p/q}\) and \(Y = W_{r/s}\) for some rationales \(p/q, r/s\).

**Remark 4.1.** For the trace identities see [Magnus, 1974].

The parameter \(c\) depends on \(\xi\). There is a correspondence between the set of cusp groups and the set of parameters \(c \in \mathbb{C}\), namely the values of \(\xi \in \mathbb{C}\), for which there is \(n \in \mathbb{N}\) such that \(t_n = \pm 2\). If the condition \(1 < \text{Im}(\xi) < 2\) is held, then the corresponding group is discrete. We have the following result which states that for this family the parameters are outside both \((M(f_\beta) \times K(f_\beta))^c\) and \((M(g_\beta) \times K(g_\beta))^c\).

**Theorem 4.2.** Let \(G_\xi = \langle A, B_\xi \rangle\) be the one-parameter family of Kleinian groups defined above. If

\[
\text{par}(G_\xi) = (\alpha, \beta(\xi), \gamma) = (0, -\xi^2 - 4, -4)
\]

then

\[
(\beta(\xi), -4) \notin (M(f_\beta) \times K(f_\beta))^c
\]

and

\[
(\beta(\xi), -4) \notin (M(g_\beta) \times K(g_\beta))^c.
\]

**Proof.** First we prove the result for the map \(f_\beta\).

Since \((\beta(\xi), -4) \in \mathbb{C} \times \mathbb{R}\) is sufficient to prove that \(\gamma = -4 \notin \text{Re}K^c(f_\beta), \forall \beta\). Let \(\beta \in \mathbb{R}, \beta \geq 0\). For \(K^c(f_\beta)\) to be connected we must have that the image, by \(f_\beta\), of the critical point \(\beta/2\) is greater than the preimage of the fixed point \(\beta + 1\). This condition gives \(-(\beta^2/4) \geq -1\), which is equivalent to \(\beta \leq 2\). If we set \(\beta \in \mathbb{R}, \beta < 0\), we get a similar condition in order for \(K^c(f_\beta)\) to be connected. In this case the condition is that the image of the critical point be greater than the preimage of the fixed point 0, which is equivalent to \(\beta \geq -4\). Then the point \((-4, -4)\) belongs to \(\partial(M(f_\beta) \times K(f_\beta))\), and for values of \(\beta \neq -4\), we have \(\gamma \neq -4\), proving the result.

For the map \(g_\beta\) with the same arguments we see that \(M(f_\beta) \cap \mathbb{R} = [-4, 2]\), and that the line \((\beta, -4)\), intersects \(M(f_\beta) \times K(f_\beta)\) only on \((-4, -4)\), and the result follows. \(\blacksquare\)

It is natural to think in a generalization of other group automorphisms.

Let \(G_\xi = \langle A, B_\xi \rangle\) as above. Let \(\psi_B\) be an automorphism in \(G\) with associated polynomial \(p_\beta\). Then \((\beta(\xi), -4) \notin (M(p_\beta) \times K(p_\beta))^c\).

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**References**


