Orbit equivalence and von Neumann algebras for expansive interval maps

C. Correia Ramos a, Nuno Martins b, Paulo R. Pinto b,*, J. Sousa Ramos b

a Department of Mathematics, Universidade de Évora, R. Romão Ramalho, 59, 7000-671 Évora, Portugal
b Department of Mathematics, Instituto Superior Técnico, Av. Rovisco Pais I, 1049-001 Lisboa, Portugal

Accepted 25 September 2006

Communicated by Prof. Ji-Huan He

Abstract

We study the orbit equivalence relation $R$, for dynamical systems $(I, \tau)$ arising from piecewise linear maps $\tau: I \to I$ on the interval $I = [0,1]$. Under regularity conditions, we prove that the crossed product von Neumann algebra $L^\infty(I) \rtimes R$ is the type III$_1$ hyperfinite factor where $\lambda \in \]0,1[\] is determined by the subgroup of $\mathbb{R}^+$ generated by $\{m(\tau(I_i))/m(I_i)\}$, with the $I_i$’s being the underlying partitioning intervals for $\tau$ and $m$ the Lebesgue measure. Thus we compute the complete invariant for the orbit structures of these maps.

1. Introduction

Von Neumann algebras has been, since its origins, a subject deeply related with quantum physics. Connes non-commutative geometry, using von Neumann algebras and $C^*$-algebras, has had a remarkable impact in both mathematics and physics, see for example [8], or [9], particularly in solid state physics and particle physics. Further relevant and interesting connections between non-commutative geometry and physics appeared in [15], like that with $\mathcal{E}^{(\infty)}$, see also [16,17]. In mathematics, the area of dynamical systems, particularly ergodic theory, has also important relations with von Neumann algebras and common developments, see [20,23] or [8].

Using von Neumann algebras as a tool borrowed from non-commutative geometry, it is our propose to study the orbit structure of a large family of interval maps arising naturally in dynamical systems. In fact, to understand the complexity of chaotic behaviour of piecewise linear maps on the interval $I = [0,1]$, we show that such maps have quite distinct orbit structures and these differences are detected by the weak equivalence which will erase many distinctions in standard ergodic theory. Thus we prove that these dynamical systems give rise to non-isomorphic hyperfinite von Neumann algebras of type III.

* Corresponding author.

E-mail addresses: ccr@uevora.pt (C. Correia Ramos), nmartins@math.ist.utl.pt (N. Martins), ppinto@math.ist.utl.pt (P.R. Pinto), sramos@math.ist.utl.pt (J. Sousa Ramos).

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doi:10.1016/j.chaos.2006.09.083
Namely, consider a map \( \tau : I \to I \) and assume there is a (finite) partition \( \{ I_1, \ldots, I_n \} \) of \( I \) with the following properties.

1. **(P1)** (Expanding piecewise linear) \( |\tau'(x)| = \beta_j > 1 \) constant for every \( x \in I_j \).
2. **(P2)** (Markov property) For every interval \( I_j \), with \( j = 1, \ldots, n \), the set \( \tau(I_j) \) is an union of intervals in \( \{ I_1, \ldots, I_n \} \).
3. **(P3)** (Aperiodicity) For every interval \( I_j \), with \( j = 1, \ldots, n \), there is a natural number \( q \) such that \( \tau^q(I_j) = I \).

If \( \tau : I \to I \) is a map satisfying (P1), (P2) and (P3), then we prove that the crossed product von Neumann algebra \( M_{\tau} = L^{\infty}(I) \times R_{\tau} \), of the abelian von Neumann algebra \( L^{\infty}(I) \) by the equivalence relation \( R_{\tau} = \{(x, y) \in I \times I : \tau^n(x) = \tau^m(y) \text{ for some } n, m \in \mathbb{N}_0 \} \)
defined by the orbits, is the hyperfinite factor of type \( III_{\lambda} \) where \( \lambda \in [0, 1] \) is determined by the multiplicative group generated by \( \{ \beta_i = m(\tau(I_i))/m(I_i) \} \) as follows:

1. If \( \log(\beta_j) / \log(\beta_i) \) is irrational for some \( i, j \), then \( L^{\infty}(I) \times R_{\tau} \) is the hyperfinite type \( III_1 \) factor.
2. If \( \log(\beta_i) / \log(\beta_j) \in \mathbb{Q} \) for all \( i, j \), then \( L^{\infty}(I) \times R_{\tau} \) is the hyperfinite type \( III_{\lambda} \) factor, where \( \lambda \) is the largest number in \( [0, 1] \) such that \( \beta_i^{-1} = 2^\lambda \) with \( m_i \in \mathbb{N} \).

Due to hyperfiniteness, different \( \lambda \)'s are attached to different orbit structures. We also compute \( \lambda \) explicitly in several examples.

The orbit equivalence for the orbit structures of a non-singular automorphism of a standard Borel space was first introduced by Dye in [4], who remarkably proved the orbit equivalence of ergodic measure preserving transformations. Later Krieger [13], using the concept of ratio set, was able to give a far reaching generalization of Dye’s theorem and thus classified up to orbit equivalence ergodic non-singular transformations.

In [10], Feldman and Moore considered in general the weak equivalence for countable standard equivalence relations on a standard Borel space \( X \) and defined the crossed product von Neumann algebras of the abelian von Neumann algebra \( L^{\infty}(X) \) by the relations without a reference to a group action on the space. It was then proved by Connes et al. in [7] that the orbit structure of a countable-to-one, non-singular endomorphism of a standard Borel space \( X \) is realized by a single non-singular automorphism, equivalently the crossed product of \( L^{\infty}(X) \) by the relation defined by the orbits is hyperfinite.

Our purpose is, by use of the above framework, to prove that the von Neumann algebra \( M_{\tau} \) is hyperfinite for the equivalence relations \( R_{\tau} \) arising from the class of dynamical systems obeying conditions (P1), (P2) and (P3), and moreover to compute the type of the von Neumann algebra \( M_{\tau} \), therefore that of the orbit equivalence. The interplay between von Neumann algebras and orbit equivalence is being developed further in the recent work of Izumi et al. [12] and by Popa [18] in the type \( II \) setting, see also [22].

The plan of the rest of the paper is as follows. In Section 2, we collect the necessary and standard background for the paper following Schmidt and Takesaki’s standard manuscripts [20,21]. It is hard to study directly the von Neumann algebra attached to the equivalence relation \( R_{\tau} \). To overcome this difficulty we will adopt, in Section 3, the following strategy: given a piecewise linear map \( \tau \) obeying conditions (P1), (P2) and (P3), we introduce another equivalence relation \( R_{\sigma} \), for the shift map \( \sigma \) defined on the subshift of finite type \( \Sigma \), associated to \( \tau \). The subshift of finite type \( \Sigma \) is defined by using a transition \( 0-1 \)-matrix \( A_{\tau} \). Then using a measure \( \rho \) on \( \Sigma \) induced by the Lebesgue measure \( m \), we can show that \( (I, R_{\sigma}, m) \) and \( (\Sigma, R_{\sigma}, \rho) \) are indeed weakly equivalent. Then we prove that the von Neumann \( L^{\infty}(\Sigma) \times R_{\sigma} \) is indeed the hyperfinite factor of type \( III_{\lambda} \). Also, the Radon–Nikodym derivative \( d \) is computable in the space \( \Sigma \), which allows us to identify the value of \( \lambda \) and thus leading us to the main result of the paper, see Theorem 5. Finally in Section 4, we apply these results in several interesting examples.

2. Preliminaries

In the sequel we follow the standard manuscripts [20,21]. Let \( (X, \mathcal{B}) \) be a standard Borel space and \( R \) be a Borel subset of \( X \times X \) which is an equivalence relation. The saturation \( R(A) \) of a subset \( A \) of \( X \) is defined by

\[ R(A) = \{ x \in X : (a, x) \in R \text{ for some } a \in A \}. \]

We will consider only such \( R \)'s with the property that \( R(\{x\}) \) is at most countable for each \( x \in A \). Now let \( m \) be a \( \sigma \)-finite measure on \( X \) with the quasi-invariant property (i.e. \( R(A) \) is \( m \)-null if \( A \) is \( m \)-null). This property depends only on the equivalence class of \( m \) with respect to absolute continuity (i.e. quasi-equivalence of measures). Given two equivalence relations as above, say \( (X_i, \mathcal{B}_i, \mathcal{M}_i, m_i) \) \((i = 1, 2)\), they are called weakly equivalent if there exist null sets \( N_i \subset \mathcal{B}_i \)
(i = 1, 2) and a non-singular Borel isomorphism \( T: X_1 \setminus N_1 \to X_2 \setminus N_2 \) such that \( T(R_i(x)) = R_i(T(x)) \) for each \( x \in X_1 \setminus N_1 \). When the above two equivalence relations are the same, the set of the non-singular Borel isomorphisms satisfying the conditions in the definition of weak equivalence forms a group, called the full group of \( (X, \mathcal{B}, R, m) \) and denoted by \([R]\). We note that the weak equivalence and the notion of the full group depend only on the class of quasi-equivalent measures. For simplicity, we only consider the equivalence relation \( R \) as above when it is ergodic: if \( A \in \mathcal{B} \) and \( R(A) \setminus A \) is m-null, then either \( A \) is m-null or \( X \setminus A \) is m-null. We are concerned with a complete invariant for the orbit equivalences of non-invertible dynamical systems on the interval \([0,1]\) equipped with the Lebesgue measure.

For this, in general, to a non-singular endomorphism \( f \) of a measure space \((X, m)\) such that \( f^{-1}(x) \) is at most countable for almost all \( x \in X \), we associate the equivalence relation \( R_f \) defined by

\[
R_f = \{(x, y) \in X \times X : f^n(x) = f^m(y), \text{ for some } n, m \in \mathbb{N}_0 \}.
\]

We remark that Renault [19] denotes \( R_f \) by \( G(X, f) \) and \( R_f \) is sometimes called the “tail equivalence with lag”. A result of [7] assures the existence of a non-singular Borel isomorphism \( T_f : X \to X \) such that \( R_f = R_{T_f} \). That is to say

\[
R_f = \{(x, T_f^n(x)) : x \in X, \ n \in \mathbb{Z} \}.
\]

2.1. Orbit structures and operator algebras

It is perhaps most natural to define von Neumann algebras as the “symmetry algebras” of unitary groups. Thus if \( H \) is a separable complex Hilbert space, von Neumann algebras \( M \subseteq \mathcal{B}(\mathcal{H}) \) are of the form \( G \) where \( G \) is a subgroup of the unitary group \( U(H) \) where the commutant \( \mathcal{S}' \) of \( \mathcal{S} \subseteq \mathcal{B}(\mathcal{H}) \) is by definition the set of operators in \( \mathcal{B}(H) \) that commute with every element in \( \mathcal{S} \). If \( \mathcal{S} \) is self adjoint containing the identity \( I_H \), i.e., \( \mathcal{S}' = \mathcal{S} \), then \( \mathcal{S}' \) coincides with the von Neumann algebra generated by \( \mathcal{S} \) which, by a theorem of Murray and von Neumann, equals the weak or strong closure of \( \mathcal{S} \). The norm closure of \( \mathcal{S} \) is a \( * \)-algebra which in general is different from \( \mathcal{S}' \). A von Neumann algebra \( M \) is called hyperfinite (or approximately finite = AF) if there exists an increasing sequence of finite \(*\)-subalgebras of \( M \) whose union generates \( M \).

Given a von Neumann algebra \( M \), its center \( \mathcal{Z}(M) \) is the von Neumann algebra \( M ' \cap M \) is an abelian von Neumann algebra; thus \( M \) is a factor if its center is the smallest possible \( \mathcal{Z}(M) = C_1 \). For instance, we obviously have \( \mathcal{Z}(L^\infty(X)) = L^\infty(X) \). Murray and von Neumann obtained the first classification of factors into types. Factors of type I \(_n \) are of the form \( \mathcal{B}(K) \) for a Hilbert space \( K \) of dimension \( n \in \mathbb{N} \cup \{ \infty \} \). A factor \( M \) is of type II \(_1 \) if \( M \) is an infinite dimensional algebra and has a normal faithful tracial state \( \tau \) such that \( \tau(\mathcal{S}) > 0 \) for \( \mathcal{S} \neq 0 \), \( \tau(ab) = \tau(ba) \), \( \tau(1) = 1 \) for all \( a, b \) in \( M \). A type II \(_\infty \) is obtained by tensoring a factor of type II \(_1 \) with \( \mathcal{B}(\ell^2(\mathbb{N})) \). Finally, \( M \) is of type III \( \lambda \) if it is not of any of the above types. Connes yielded in [5] a finer classification of type III factors into type III \( \lambda \) where \( \lambda \in [0, 1] \) by letting

\[
S(M) = \bigcap_{\phi \text{faithful normal weight}} \text{Sp}(\Delta_\phi),
\]

where \( \Delta_\phi \) is the Tomita–Takesaki modular operator and \( \text{Sp}(\Delta_\phi) \) its spectrum, see e.g. [21]. Then Connes proved that \( S(M) \setminus \{0\} \) is a closed subgroup of the positive real numbers and naturally arrive at the following finer classification for type III factors:

- Type III \(_0 \) if \( S(M) = \{0\} \cup \{1\} \)
- Type III \(_1 \) for \( \lambda \in [0, 1[ \), if \( S(M) = \{0\} \cup \{\lambda^2\} \)
- Type III \(_1 \) if \( S(M) = [0, +\infty[ \).

We further remark that if \( M \) is not a type III factor then \( S(M) = \{1\} \).

In the sequel we will consider von Neumann algebras via the crossed product construction by an \( \mathbb{Z} \) action. For that we start with the equivalence relation \( R_f \) associated to the map \( f : X \to X \) and denote by \( T_f : X \to X \) the underlying isomorphism of the measure space \((X, m)\). Then we can define an action of the discrete abelian group \( \mathbb{Z} \) on \( L^\infty(X) \) as follows:

\[
(x_\xi)(x) = \xi(T_f^nx), \quad n \in \mathbb{Z}, \quad \xi \in L^\infty(X).
\]

Next there are unitaries \( u_n \) implementing the action: \( x_\xi = u_\xi u_n^* \), see e.g. [21]. Finally, the crossed product \( L^\infty(X) \times_{\mathbb{Z}} \mathbb{Z} \) is the von Neumann algebra generated by \( L^\infty(X) \) and the unitaries \( \{u_n\} \)'s. So that

\[
x = \sum_{n \in \mathbb{Z}} \xi_n u_n,
\]

with \( \xi_n \in L^\infty(X) \) and \( A \) is a finite subset of \( \mathbb{Z} \), is a typical element in the crossed product algebra. As a way of example, if we let \( X = \{x\} \), then \( L^\infty(X) \cong \mathbb{C} \) and the crossed product \( \mathbb{C} \times \mathbb{Z} \) is the group algebra of \( \mathbb{Z} \).
We also note that if \( R_f \) is ergodic, then \( T_f \) is ergodic and free, hence the crossed product von Neumann algebra \( M_f = L^\infty(X) \rtimes_{\varepsilon} \mathbb{Z} \) is in this case a hyperfinite factor. Thanks to Krieger’s theorem [13, Thm. 8.4] on ergodic flows, this crossed product algebra does not depend on the choice of \( T_f \), hence we denote \( L^\infty(X) \rtimes_{\varepsilon} \mathbb{Z} \) simply by \( L^\infty(X) \times R_f \). Furthermore, for two endomorphisms \( f_1, f_2 \) of \( X \) such that \( R_{f_1} \) and \( R_{f_2} \) are both ergodic, \( R_{f_1} \) and \( R_{f_2} \) are weakly equivalent if and only if \( L^\infty(X) \times R_{f_1} \) is \( 
abla \)-isomorphic to \( L^\infty(X) \times R_{f_2} \). At this point we emphasize that the hyperfiniteness of the algebras is quite crucial to determine the complete invariant for the weak equivalence classes (for instance, see [11] when the underlying von Neumann algebras are not hyperfinite or [22] and references therein for further explanations).

For the class of von Neumann algebras arising from equivalence relations \( R \), it should be noted that the crossed product \( L^\infty(X) \times R \) depends only on the quasi-equivalent class of \( m \). Assuming that \( R \) is ergodic, \( L^\infty(X) \times R \) is a type III factor if and only if there is no absolutely continuous \( \sigma \)-finite invariant measure \( m_\sigma \) with respect to \( m \) (see [10] or [21]). In this case, Connes’ \( S \) invariant of the algebra \( L^\infty(X) \times R \) coincides with the Krieger invariant

\[
\left\{ \lambda \in R^+_0 : \text{for all } \epsilon > 0 \text{ and } E \subset \mathcal{B} \text{ with } m(E) > 0 \text{ there exists } F \subset \mathcal{B} \text{ with } m(F) > 0, F \subset E \text{ and } T \in [R] \text{ such that } \left| \frac{d \mu \circ T}{dm}(x) - \lambda \right| < \epsilon \text{ for almost all } x \in F \text{ and } T(F) \subset E \right\}.
\]

The left counting measure \( \nu_l \) (resp. the right counting measure \( \nu_r \)) on \((R, R \cap (\mathcal{B} \times \mathcal{B}))\) is defined by

\[
\nu_l(C) = \int \text{Card}(\pi_l^{-1}(x) \cap C) \, dm(x),
\]

(resp. \( \nu_r(C) = \int \text{Card}(\pi_r^{-1}(x) \cap C) \, dm(x) \))

for each \( C \in R \cap (\mathcal{B} \times \mathcal{B}) \) and the Radon–Nikodym cocycle \( d_m \) is defined as the function on \( R \) given by \( d_m(x, y) = \frac{d \mu(x,y)}{\nu_l(x,y)} \). Here \( \pi_l \) (resp. \( \pi_r \)) is the map from \( X \times X \) to \( X \) such that \( \pi_l(x,y) = x \) (resp. \( \pi_r(x,y) = y \)).

Thus Krieger’s set is also expressed in terms of \( d_m \) as the essential range \( r(d_m) \), namely \( S(M) = \{ \lambda \in R^+_0 : \text{for all } \epsilon > 0 \text{ and } E \subset \mathcal{B} \text{ with } m(E) > 0 \text{ such that } m(E \setminus \{ x \in E : d_m(x,y) \in (\lambda - \epsilon, \lambda + \epsilon) \text{ for some } y \in E \}) = 0 \} \) as in e.g. [10, Proposition 8.4].

3. Symbolic dynamics and von Neumann algebras

Let \( I = [0,1] \) and \( m \) be the Lebesgue measure. Consider a map \( \tau : I \to I \) and a partition \( \{I_1, \ldots, I_n\} \) of \( I \) \((\bigcup_{j=1}^n I_j = I)\) with the following properties.

**P1** (Expanding piecewise linear) \( |\tau'(x)| = \beta_j > 1 \) constant for every \( x \in I_j \).

**P2** (Markov property) For every interval \( I_j \) with \( j = 1, \ldots, n \), the set \( \tau(I_j) \) is an union of intervals in \( \{I_1, \ldots, I_n\} \).

**P3** (Aperiodicity) For every interval \( I_j \) with \( j = 1, \ldots, n \), there is a natural number \( q \) such that \( \tau^q(I_j) = I \).

The subintervals \( I_j \)'s can be obtained from the iterates of the critical points of \( \tau \). If \( \tau \) satisfies (P2) then the orbits of its critical points are necessarily finite. From property (P3) the map \( \tau \) is exact, i.e., \( \lim_{k \to \infty} m(\tau^k(E)) = 1 \) for every Borel set \( E \) with positive measure. In fact it satisfies a stronger property: for every Borel set \( E \) of positive measure there is a natural number \( k \) so that \( \tau^k(E) = I \), see [1]. The property (P3) may seem a very strong condition but directly implies ergodicity and at the same time excludes some transient behaviour of the map which has no role in our approach.

We associate to the partition \( \{I_1, \ldots, I_n\} \) the Markov transition matrix \( A_t = (a_{ij})_{1 \leq i,j \leq k} \in \text{Mat}_k(\{0,1\}) \) with

\[
a_{ij} = \begin{cases} 1 & \text{if } \tau(I_j) \supset I_i, \\ 0 & \text{otherwise}. \end{cases}
\]

From property (P3) there is a natural number \( k \) so that every entry in \( A_t \) is positive (the matrix \( A_t \) is aperiodic).

Let \( (\Sigma, \sigma) \) denote the subshift of finite type associated to \( \tau \), the partition \( \{I_1, \ldots, I_n\} \), and to the transition matrix \( A_t \).

The set \( \Sigma_t \subset \{1, \ldots, n\}^\mathbb{N} \) is defined by

\[
\Sigma_t = \{(s_i)_{i \in \mathbb{N}} : a_{s_{i+1}s_i} = 1, i \in \mathbb{N}\},
\]

and the map \( \sigma \) is defined by \( \sigma(s_i)_{i \in \mathbb{N}} = (s_{i+1})_{i \in \mathbb{N}} \). Observe that \( \sigma(\Sigma_t) = \Sigma_t \).
Let \( c_0 < c_1 < \cdots < c_n \) denote the border points of the partition \( \{I_1, \ldots, I_n\} \). The \textit{address map} 
\[ ad : I \rightarrow \{1, 2, \ldots, n, C_0, \ldots, C_n\} \]
is defined by
\[
ad(x) = \begin{cases} 
  i & \text{if } x \in I_i, \\
  C_j & \text{if } x = c_j.
\end{cases}
\]  
(3)

Note that the elements in \( \{1, 2, \ldots, n, C_0, \ldots, C_n\} \) are seen as formal objects.

The \textit{itinerary map} is defined by
\[
\it(x) = (ad(x)ad(\tau(x))ad(\tau^2(x)) \ldots)
\]
We say that \( s_0 \cdots s_{n-1} \in \{1, \ldots, n\}^n \) is an \textit{admissible word} with respect to \( \tau \) (or \( \Sigma_\tau \)) if it occurs in some sequence of \( \Sigma_\tau \).

Let \( I_0 := I \setminus \bigcup_{k=0}^{\infty} \tau^{-k}\{c_0, \ldots, c_n\} \).

Let \( \varphi \) be the map defined as the restriction \( \varphi|_{I_0} \), so that \( \varphi(I_0) = \Sigma_\tau \). Thus 
\[
\varphi(\tau|_{I_0}(x)) = \sigma \circ \varphi(x),
\]
i.e. \( (I_0, \tau|_{I_0}) \) and \( (\Sigma_\tau, \sigma) \) are topologically conjugated. See [1] or [23] for further details.

For an admissible word \( s_0 \cdots s_k \) we define the cylinder set \([s_0 \cdots s_k]\) by
\[
[s_0 \cdots s_k] = \{(t)_i \in C_0 \in \Sigma_\tau : t_0 = s_0, \ldots, t_k = s_k\}.
\]

In the same way the cylinder \( I_{s_0 \cdots s_k} \) is defined as the set of points in \( I_0 \) whose itineraries have as prefix the admissible word \( s_0 \cdots s_k \).

Observe that with these definitions \( \varphi(I_{s_0 \cdots s_k}) = [s_0 \cdots s_k] \).

Since 
\[
I_{s_0 \cdots s_k} = \bigcap_{i=0}^{n} \tau^{-i}(I_{s_k}) = I_{s_0} \cap \tau^{-1}(I_{s_1}) \cap \cdots \cap \tau^{-n}(I_{s_k})
\]
then for any cylinder \( I_{s_0 \cdots s_k} \)
\[
\tau(I_{s_0 \cdots s_k}) = I_{s_1 \cdots s_k}\.
\]
The dynamical systems \( (I, m, \tau) \) and \( (I_0, m, \tau|_{I_0}) \) are isomorphic since \( I = I_0 \) m-a.e. On the other hand \( \varphi \) in \( I_0 \) is invertible (and measurable). Defining \( \rho := m \circ \varphi^{-1} \) the dynamical systems \( (I_0, m, \tau) \) and \( (\Sigma_\tau, \rho, \sigma) \) are isomorphic and then \( (I, m, \tau) \) is isomorphic to \( (\Sigma_\tau, \rho, \sigma) \). We have, naturally,
\[
\rho([s_0 \cdots s_k]) = m(I_{s_0 \cdots s_k}).
\]

**Lemma 1.** The measure \( m \) satisfies \( m(I_{s_0 \cdots s_k}) = \beta_{s_0}^{-1} \beta_{s_1}^{-1} \cdots \beta_{s_k}^{-1} m(I_{s_k}) \). Thus
\[
\rho([s_0 \cdots s_k]) = \beta_{s_0}^{-1} \beta_{s_1}^{-1} \cdots \beta_{s_k}^{-1} \rho([s_k]).
\]

**Proof.** From property (P1) we have \( |\tau(\varphi(x))| = \beta \). Then for every \( E \subset I_0 \) measurable, we have, \( m(\tau(E)) = \beta m(E) \). Since \( \tau(I_{s_0 \cdots s_k}) = I_{s_1 \cdots s_k} \) and \( I_{s_0 \cdots s_k} \subset I_{s_k} \) we have
\[
m(\tau(I_{s_0 \cdots s_k})) = \beta_{s_0} m(I_{s_0 \cdots s_k}) \quad \text{and} \quad m(I_{s_1 \cdots s_k}) = \beta_{s_0} m(I_{s_0 \cdots s_k}).
\]
Repeating the same procedure we will get $m(I_{s_0}) = \beta_{s_0} \cdots \beta_{s_n} m(I_{s_n})$.
Since $\rho$ is induced by $m$ we have
\[
\rho(s_0 \cdots s_k) = m(I_{s_0 \cdots s_k}) = \beta_{s_0}^{-1} \cdots \beta_{s_k}^{-1} \beta_{s_0}^{-1} \cdots \beta_{s_n}^{-1} \rho([s_k]). \quad \square
\]

Besides the equivalence relation $R$, we now also have the equivalence relation $R_\sigma$ defined by the shift map on $\Sigma$:
\[
R_\sigma = \{(s, t) : s, t \in \Sigma, \sigma^k(s) = \sigma^k(t), \text{ for some } n, m \in \mathbb{N}_0\}.
\]

Since $(I, m, \tau)$ and $(\Sigma, \rho, \sigma)$ are isomorphic then $R_\sigma$ is weakly equivalent to $R_\sigma$. Thus the von Neumann algebras $L^\infty(I, m) \times R_e$ and $L^\infty(\Sigma, \rho) \times R_\sigma$ are *-isomorphic.

**Lemma 2.** The relation $R_\sigma$ is ergodic with respect to the measure $\rho$.

**Proof.** The relation $R_\sigma$ is ergodic with respect to $\rho$ if whenever $R_\sigma(dE) = E, \rho$-a.e. then $\rho(E) = 1$ or $\rho(E) = 0$. From the assumed conditions for the map $\tau$ we know that $A_z$ is aperiodic. In this case for every Borel set $E$, with $\rho(E) > 0$ we have that there is a natural number $q$ so that $\rho(\sigma^q(E)) = 1$. Since $\sigma^q(E) \subset R_\sigma(E)$ for every $k \geq 0$, and $\sigma$ is non-singular we have $\rho(R_\sigma(E)) = 0$ if $\rho(E) = 0$ or $\rho(R_\sigma(E)) = 1$ if $\rho(E) > 0$. Thus $R_\sigma$ is ergodic. \(\square\)

**Lemma 3.** The von Neumann algebra $L^\infty(I, m) \times R_e \cong L^\infty(\Sigma, \rho) \times R_\sigma$ is an hyperfinite type III factor.

**Proof.** The algebra $L^\infty(I, m) \times R_e$ is *-isomorphic to $L^\infty(\Sigma, \rho) \times R_\sigma$ and $L^\infty(\Sigma, \rho) \times R_\sigma$ is a factor since $R_\sigma$ is an ergodic equivalence relation from Lemma 2. It is hyperfinite from [7].

Now we prove that it is of type III. Let $\mu$ be an invariant measure for $R_e$ which is absolutely continuous with respect to $\rho$ and consider $s$ fixed. Then by property (P3) we can find $s_1 \cdots s_k$ and $t_1 \cdots t_k$ such that $s_j \neq = t_j$ for some $j$ and moreover $ss_1 \cdots s_k$ and $st_1 \cdots t_k$ are admissible words. Let $P_1 = [ss_1 \cdots s_k]$ and $P_2 = [st_1 \cdots t_k]$. The maps $\phi_i := \sigma_i^{k+1}$, $i = 1, 2$ are partial isomorphisms with domain $P_i$ and range $[s]$, and the graph of $\phi_i$ satisfies $\{(x, \phi_i(x)) \in \Sigma\}$ by [10] a measure $\rho$ is invariant for the equivalence relation $R_\sigma$ if and only if it is invariant for every partial isomorphism whose graph is contained in $R_\sigma$. If $\mu$ is invariant then it is invariant for each $\phi_i$, and in particular $\mu(P_1) = \mu\phi_i^{-1}([s]) = \mu([s])$. On the other hand, since $P_1 \cup P_2 \subset [s], P_1 \cap P_2 = \emptyset$, we have $\mu([s]) \geq \mu(P_1) + \mu(P_2) = 2\mu([s])$. Then $\mu([s]) = 0$ or $\infty$. Hence $\mu$ takes 0 or $\infty$ on any cylinder set, outer regularity of $\mu$ implies $\mu \equiv 0$ or $\mu \equiv \infty$. So there is no nonzero $R_\sigma$ invariant measure $\mu \neq \rho$. Thus $L^\infty(\Sigma, \rho) \times R_e$ is a type III factor. \(\square\)

Now we denote the left and right counting measures on $R_e$ by $\nu_l$ and $\nu_r$, and also denote the Radon–Nikodym cocycle $dv_l/dv_r$ by $d\nu$.

**Lemma 4.**

1. We have $d\nu((s_1), \sigma^k(s_1)) = \Pi_{i=0}^{n-1} \beta_1^{-1}$ for each $(s_1) \in \Sigma$.
2. We further have that $\beta_i \in S(L^\infty(I) \times R_\sigma)$ for each $i$.

**Proof.** Let $\phi := \sigma_i^{k+1} : [s_0 \cdots s_n] \to [s_n]$. Then $\phi$ is a non-singular Borel isomorphism. By definition of $\rho$ and Lemma 1, we have $d\phi^{-1}(x) = \Pi_{i=0}^{n-1} \beta_i^{-1}$ for any $x \in [s_n]$. By [10, Proposition 2.2] we conclude part (1) of the lemma.

Let us prove part (2). Suppose there is a Borel set $E$ such that $\rho(E) > 0$ and for some $i$, $\beta_i$ does not belong to the essential range of $d\phi|_E$, i.e. $\beta_i \notin r(d\phi|_E)$. Since by Lemma 3, $L^\infty(I) \times R_e$ is a factor of type III, there exists a non-singular Borel isomorphism $\phi : \Sigma \to E$ such that the graph of $\phi$ lives inside $R_e$, i.e. $\{(x, \phi(x)) : x \in \Sigma \} \subset R_e$. Set $\tilde{d}(x, y) = d\phi(x, \phi(y))$. Then thanks to [10, Proposition 7.5], $\tilde{d}$ is cohomologous to $d\rho$, hence $\beta_i \notin r(d\rho|_E) = r(d\phi)$. However, this contradicts the first part of this lemma. Therefore, we have proven part (2) of the lemma. \(\square\)

We are ready to state and prove the main result of the paper.

**Theorem 5.** Let $\tau : I \to I$ so that there is a partition $\{I_1, \ldots, I_n\}$ of $I$, satisfying conditions (P1), (P2) and (P3). Let $[\tau_i^j(x)] = \beta_j$, then

1. if $\log(\beta_i) \log(\beta_j)$ is irrational for some $i, j$, then $L^\infty(I) \times R_e$ is the hyperfinite type III1 factor;
2. if $\log(\beta_i) \log(\beta_j) \in \mathbb{Q}$ for all $i, j$, then $L^\infty(I) \times R_e$ is the hyperfinite type III1 factor, where $\lambda$ is the largest number in $\{i \in \mathbb{N} : \text{such that } \beta_i^\lambda = \lambda^m \text{ with } m \in \mathbb{N}\}$. 

**Proof.** First note that by Lemma 4 and [10, Proposition 8.5], $S(L^\infty(I) \times R_\pi) \setminus \{0\}$ is a closed subgroup of $\mathbb{R}_+$ generated by the $\beta_i$’s. In case $\log(\beta_i)/\log(\beta_j)$ is irrational for some $i,j$, Lemma 4 implies that $S(L^\infty(I) \times R_\pi) \setminus \{0\} = [0, +\infty]$. Otherwise $S(L^\infty(I) \times R_\pi) = \{\lambda^x\} \cup \{0\}$. The value for $\lambda$ is the largest number in $]0, 1[$ such that $\beta_i^{-1} = \lambda^m$ with $m_i \in \mathbb{N}$. In fact let $\beta_i^{-1} = e^a$ for $i = 1, \ldots, n$. Considering $i = 1$ fixed, we have $\log(\beta_i)/\log(\beta_1) = a/\alpha_1 = p/q$ with $p_j, q_j$ natural numbers such that $\gcd(p_j, q_j) = 1$. Consider the product

$$\beta_1^{a_1} \beta_2^{a_2} \cdots \beta_n^{a_n} = \exp(k_1 \alpha_1 + k_2 \alpha_2 + \cdots + k_n \alpha_n) = \exp \left( \frac{q_1}{q_n} (k_1 q_2 \cdots q_n + \cdots + k_n p_2 \cdots q_{n-1}) \right)$$

for $k_1, \ldots, k_n \in \mathbb{Z}$. The natural numbers $\{q_2, q_3, \ldots, p_n q_2 \cdots q_{n-1}\}$ generate $\mathbb{Z}$ as an additive group if and only if $\gcd(q_2, q_3, \ldots, p_n q_2 \cdots q_{n-1}) = 1$. In that case $\lambda = \exp \frac{a}{\alpha_1}$. If not we have

$$\bar{\lambda} = \exp \frac{-a_1}{q_2 \cdots q_n \gcd(q_2, q_3, \ldots, p_n q_2 \cdots q_{n-1})}$$

and $\bar{\lambda}$ is then the maximal number in $]0, 1[$ with $\beta_i^{-1} = \bar{\lambda}^m$ and $s = q_2, q_3, \ldots, \gcd(q_2, q_3, \ldots, p_n q_2 \cdots q_{n-1})$ a natural number. The same procedure can be followed for any $i = 1, \ldots, n$ obtaining the same conclusion. □

**Remark 6.** For the particular case in which $\tau_\beta$ is the (unimodal) tent map, where $\beta$ is the absolute value of the slope, the algebra $M_{\tau_\beta}$ is the hyperfinite factor of type $III_{1/\beta}$, which was claimed in [2]. See Example 8.

### 4. Examples

**Example 7.** Let $1 < \beta < 2$ and $\tau_\beta$ be the map

$$\tau_\beta(x) = \beta x \pmod{1}.$$  

The associated partition is $\{I_1, I_2, \ldots, I_n\}$ with $n = \lfloor \beta \rfloor$ the integer part of $\beta$ and

$$I_j = [(j-1)\beta^{-1}, j\beta^{-1}], \quad j = 1, \ldots, n-1 \quad \text{and} \quad I_n = [(n-1)\beta^{-1}, 1].$$

The value $\beta$ is such that the orbit of 1 is a finite set. In that case the property (P2) is fulfilled. Then $S(M_{\tau_\beta}) = \{\bar{\lambda}^x\} \cup \{0\}$, with $\bar{\lambda} = \beta^{-1}$, and $M_{\tau_\beta}$ is the hyperfinite factor of type $III_{1/\beta}$. Let $\beta = \sqrt{3} + 1$. The graph of $\tau_\beta$ is presented in Fig. 1. In this case we have

$$I_1 = [0, (\sqrt{3} + 1)^{-1}], \quad I_2 = [(\sqrt{3} + 1)^{-1}, 2(\sqrt{3} + 1)^{-1}], \quad I_3 = [2(\sqrt{3} + 1)^{-1}, 1].$$

The transition matrix is given by

![Fig. 1. Graph of $\tau_\beta$ with $\beta = 1 + \sqrt{3}$.](image-url)
The algebra $M_{\hat{s}b}$ is the hyperfinite factor of type $\text{III}_1$ with $\hat{\lambda} = \beta^{-1} = \frac{\sqrt{3}-1}{2}$.

**Example 8.** Let $\tau_{\beta_1, \beta_2}$ be the map

$$
\tau_{\beta_1, \beta_2} = \begin{cases} 
\beta_1 x & \text{if } 0 \leq x < \beta_1^{-1}, \\
\beta_2(x - \beta_1^{-1}) & \text{if } \beta_1^{-1} \leq x < 1.
\end{cases}
$$

The associated partition is $\{I_1, I_2\}$ with $I_1 = [0, \beta_1^{-1}], I_2 = [\beta_1^{-1}, 1]$ and the transition matrix is

$$
A_{\tau_{\beta_1, \beta_2}} = \begin{pmatrix} 1 & 1 \\
1 & 1 \end{pmatrix}.
$$

Let $x_j = \log \beta_j$ and consider the general product

$$
\beta_1^{k_1} \beta_2^{k_2} = e^{x_1 k_1 + x_2 k_2},
$$

where $x_1/x_2 = p/q \in \mathbb{Q}$ with $p, q$ natural numbers satisfying $\gcd(p, q) = 1$. In this case

$$
e^{x_1 k_1 + x_2 k_2} = e^{x_1 (k_1 p + k_2 q)} = e^{x_2 (k_1 p + k_2 q)}.
$$

Since $\gcd(p, q) = 1$ the additive group generated by $(p, q)$ is $\mathbb{Z}$. Then $S(M_{\tau_{\beta_1, \beta_2}}) = \{\lambda \mathbb{Z}\} \cup \{0\}$, with $\lambda = e^{\frac{x_1}{x_2}} = e^{\frac{x_2}{x_1}}$, and $M_{\tau_{\beta_1, \beta_2}}$ is the hyperfinite factor of type $\text{III}_1$.

Suppose $x_1/x_2 = \theta \notin \mathbb{Q}$. In this case

$$
e^{x_1 k_1 + x_2 k_2} = e^{x_2 (k_1 \theta + k_2)}.
$$

The additive group generated by $(1, \theta)$ is dense in $\mathbb{R}$. Thus the closure of $\{\lambda \mathbb{Z} + \theta \mathbb{Z}\} \cup \{0\}$ is $[0, +\infty[$ and $M_{\tau_{\beta_1, \beta_2}}$ is the hyperfinite factor of type $\text{III}_1$.

In Fig. 2 we show the graph of the map $\tau_{\beta_1, \beta_2}$ with $\beta_1 = 3$ and $\beta_2 = 3/2$. In this case

$$
\frac{\log \beta_1}{\log \beta_2} = \frac{\log 3}{\log 3 - \log 2} \notin \mathbb{Q}.
$$

Thus $M_{\tau_{\beta_1, \beta_2}}$ is the hyperfinite factor of type $\text{III}_1$. Note that we obtain the same result if the map is a tent map type

$$
\tau_{\beta_1, \beta_2} = \begin{cases} 
\beta_1 x & \text{if } 0 \leq x < \beta_1^{-1}, \\
\beta_2(1 - x - \beta_1^{-1}) & \text{if } \beta_1^{-1} \leq x < 1.
\end{cases}
$$

![Fig. 2. Graphs of the map $\tau_{\beta_1, \beta_2}$ and $\tau_{\hat{s}b, \beta_2}$, for $\beta_1 = 3$ and $\beta_2 = 3/2$.](image)
since $M_{\tau_1, \tau_2} \cong M_{\tau_1, \tau_2}$. See Fig. 2 for the graph of $\tau_{\beta_1, \beta_2}$ with $\beta_1 = 3$ and $\beta_2 = 3/2$.

**Remark 9.** With expansive maps $\tau$ we can never yield a type III$_0$ factor for this class of dynamical systems.

**Acknowledgements**

First author acknowledges CIMA-UE for financial support. The other authors were partially supported by FCT/POCTI/FEDER.

**References**


[12] Popa S. Strong rigidity of II$_1$ factors arising from malleable actions of w-rigid groups, II. Invention Math 2006;165:369–408.


