Existence and regularity for scalar minimizers of affine nonconvex simple integrals

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Abstract

Existence of AC minimizers $x : [a, b] \rightarrow \mathbb{R}$ is proved for the nonconvex integral $\int_a^b \{ \rho(x) h(x') + \varphi(x) \} \, dt$, under the general hypotheses of lower semicontinuity, boundedness below, and superlinear growth at infinity in $x'(\cdot)$. Any nonconvex function $h : \mathbb{R} \rightarrow [0, +\infty]$ will do, provided it is convex at $\zeta = 0$.

Moreover, minimizers are shown to satisfy several regularity properties, under adequate hypotheses.

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To prove existence of minimizers, for general integrals of absolutely continuous functions $x(\cdot)$, it is usual to impose convexity, together with superlinear growth at infinity, on the integrand relative to the velocities $x'(\cdot)$; and then apply the direct method (see e.g. [8–10]). However, for integrals which may be written as a sum,

$$\int_a^b \{ h(x'(t)) + \varphi(x(t)) \} \, dt,$$

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existence has been proved also for nonconvex \( h(\cdot) \) (see \([1,2,5–7,12–15,20]\)). Such results have been obtained by considering the convexified integral, (1) with \( h(\cdot) \) replaced by its bipolar—or closed convex hull—\( h^{**}(\cdot) \), and by imposing the usual hypotheses under which the direct method yields minimizers for such convex integral. These are called relaxed minimizers of the nonconvex integral. One then proves existence, of true minimizers for (1), replacing convexity on \( h(\cdot) \) by adequate compensating assumptions on \( \varphi(\cdot) \), like concavity (or monotonicity, or a mixture of both, in case \( x(\cdot) \) is scalar).

Another possibility is to impose on \( h(\cdot) \) only an extremely weak hypothesis of convexity: to be convex at the point \( \xi = 0 \) only, i.e.

\[
h^{**}(0) = h(0).
\]

In this case, the hypotheses on \( \varphi(\cdot) \) may be very much relaxed. Indeed, in [1] existence was proved, for (1) under (2), only asking the set of strict local minimum points of \( \varphi(\cdot) \) to be locally finite. And later, in [11], the same result was obtained by imposing on \( \varphi(\cdot) \) only to have level sets with null boundary (e.g. countable boundary, as is true in case \( \varphi(\cdot) \) is any combination of elementary functions). Still in [11], even this condition was dropped for more special \( h(\cdot) \) (with at most two intervals of nonconvexity, both starting at \( \xi = 0 \)). This question of existence of minimizers, for (1) under (2), was finally settled in [16], proving existence without any extra condition apart from lower semicontinuity, boundedness below and superlinear growth at infinity, i.e.

\[
\frac{h(\xi)}{|\xi|} \to +\infty \quad \text{as } |\xi| \to \infty.
\]

A continuation of this research deals with more general integrals, of “affine” type:

\[
\int_a^b \{\rho(x(t))h(x'(t)) + \varphi(x(t))\} \, dt.
\]

Very special cases of this integral had been treated already in [11]. The first case was \( \varphi(\cdot) = \text{constant} \) and level sets of \( \rho(\cdot) \) with null boundary. In the second, \( h^{**}(\cdot) \) was asked to have at most two intervals of nonconvexity satisfying a special condition, relative to the level sets of \( \varphi(\cdot) + q\rho(\cdot) \): to have null boundary for adequate \( q \). Existence of minimizers for (4) with more general nonconvex \( h(\cdot) \) satisfying (2) was then proved in [19], assuming the level sets

\[
\{s \in \mathbb{R} : \varphi(s) + q \rho(s) = \text{constant}\}
\]
to have null boundary, at least for suitably defined numbers \( q \). Existence results prior to [19] considered \( h(\cdot) \) only with finite values.

The purpose of this paper is to prove existence of minimizers for the nonconvex integral (4) without using any such special assumptions. Indeed, we assume only the usual basic hypotheses which are needed to obtain existence of minimizers for the convexified integral

\[
\int_a^b \{\rho(x(t))h^{**}(x'(t)) + \varphi(x(t))\} \, dt,
\]
by the standard direct method. The proof constructs a true minimizer (from a relaxed minimizer satisfying the monotonicity properties proved in [17]) by applying the Liapunov theorem on the range of vector measures (see e.g. [4]). However, the application of this theorem is not straightforward: integrability difficulties had to be surpassed, by applying Liapunov infinitely many times. As compared with the “sum case” $\rho(\cdot) \equiv 1$ of [16], and with the “affine regular case” of [19], these difficulties were raised by the weakness of the hypotheses desired. Namely: to assume nothing (unlike in [16]) on the level sets of $\varphi(\cdot)$, to allow $h^{**}(\cdot)$ (unlike in [16]) to take $+\infty$ values, to have domain $(h^{**})^{-1}(\mathbb{R})$ not open, and to be infinitely steep at $\zeta = 0$; to impose on $\rho(\cdot)$ no restrictions concerning local boundedness above (allowing it to be, e.g., not in $L^1_{\text{loc}}$).

It is shown also that the minimizer here constructed, for the nonconvex integral, satisfies several regularity properties (Lipschitz continuity, piecewise differentiable continuity, piecewise affinity) under appropriate hypotheses, besides satisfying the necessary condition of DuBois–Reymond (the differential inclusion (6)), in case the domain $(h^{**}(\cdot))^{-1}(\mathbb{R})$ is open.

Here is the main result:

**Theorem 1.** Let $\rho : \mathbb{R} \to [1, +\infty)$, $h : \mathbb{R} \to [0, +\infty]$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be lower semicontinuous functions, bounded below.

Assume $h(\cdot)$ to satisfy the conditions (2), of convexity at $\zeta = 0$, and (3), of superlinear growth at infinity.

Then for any $A, B$ the nonconvex integral (4) has a minimizer $x(\cdot)$—among the absolutely continuous functions with graph passing through the points $(a,A), (b,B)$.

**Corollary 2.** Under the same hypotheses of Theorem 1, the minimizer $x(\cdot)$ may be chosen so as to satisfy, moreover, the following regularity properties:

(i) there exists an interval $[a', b']$ along which $x(\cdot)$ is constant, and $x(\cdot)$ is strictly monotone along each of the remaining subintervals, $[a, a']$ and $[b', b]$;

(ii) if the quotient $\varphi/\rho(\cdot)$ is locally bounded above then $x(\cdot)$ is Lipschitz continuous;

(iii) if the domain $(h^{**}(\cdot))^{-1}(\mathbb{R})$ is open and the minimum is finite, then $x(\cdot)$ satisfies the DuBois–Reymond differential inclusion: there exists a constant $c$ such that

$$h(x'(t)) \leq \frac{c - \varphi(x(t))}{\rho(x(t))} + x'(t)\varphi(x(t))^{**} \quad \text{a.e. on } [a, b];$$

(iv) if, in addition to the extra hypotheses stated in (ii) and (iii), the functions $\varphi(\cdot)$ and $\rho(\cdot)$ are (locally) piecewise continuous, the level sets of $\varphi(\cdot) + q\rho(\cdot)$ (for $q \in \mathbb{R}$) are (locally) finite, and $\text{epi } h^{**}(\cdot)$ has finitely many 1-dim faces and nonsmooth vertices (over each bounded interval) then $x(\cdot)$ is piecewise-$C^1$;

(v) if, moreover, $h^{**}(\cdot)$ is (locally) piecewise affine then $x(\cdot)$ is piecewise affine.

**Remark 1.** The extra hypothesis, in (iii), of the domain $(h^{**}(\cdot))^{-1}(\mathbb{R})$ to be open is not necessary if, for some reason, it can be guaranteed that, for some minimizer $y(\cdot)$ of the convexified integral (5),

$$y'(t) \in \text{interior}[(h^{**}(\cdot))^{-1}(\mathbb{R})] \quad \text{a.e.}$$
The extra hypotheses of (iv) are satisfied by the functions \( \varphi(\cdot), \rho(\cdot), h(\cdot) \) usually appearing in real-world applications, at least when \( h(\cdot) \) takes only finite values—e.g. whenever these functions are piecewise analytic over each bounded interval.

\( \partial h^{**}(\cdot) \) is the subdifferential of the convex lower semicontinuous function \( h^{**}(\cdot) \), the bipolar, or closed convex hull, of \( h(\cdot) \) (see e.g. [10,21]).

A point \((\xi, h^{**}(\xi))\) is called a nonsmooth vertex of \( \text{epi } h^{**}(\cdot) \) if \( \partial h^{**}(\xi) \) has more than one point. A 1-dim face of \( \text{epi } h^{**}(\cdot) \) is the intersection of \( \text{epi } h^{**}(\cdot) \) with one of its supporting straight lines, having more than one point.

By locally piecewise affine (resp. locally piecewise continuous) we mean a function satisfying the following property: each bounded interval has a partition by finitely many points, such that the function is affine (resp. uniformly continuous) inside each open subinterval of the partition. And piecewise-C\(^1\) means to have a piecewise continuous derivative.

**Proof.** (a) The convexified integral (5) has a minimizer \( y(\cdot) \) satisfying the properties stated, for \( x(\cdot) \), in Corollary 2, together with

\[
y'(t) \notin \text{interior}[\left(\partial h^{**}(\cdot)\right)^{-1}(\partial h^{**}(0))] \]

a.e. in \( [a,a'] \cup [b',b] \), by Ornelas [18, Theorems 1, 3]. In particular, we may assume

\[
h(\xi) = h^{**}(\xi) \quad \forall \xi \in \left(\partial h^{**}(\cdot)\right)^{-1}(\partial h^{**}(0)) := \{\xi \in \mathbb{R} : \partial h^{**}(\xi) \cap \partial h^{**}(0) \neq \emptyset\}.
\]

This proof will then consist in modifying \( y(\cdot) \) on the intervals \( [a,a'], [b',b] \), along each of which \( y(\cdot) \) is strictly monotone, in order to construct the desired minimizer \( x(\cdot) \) of the nonconvex integral. We may therefore assume \( y(\cdot) \) to be, say, strictly increasing along \( [a,b] \). (If \( y(\cdot) \) strictly decreased along \( [a,b] \), the reasoning would be similar.)

Clearly, there exist at most countably many numbers \( m_r \in \mathbb{R} \) such that

\[
0 < z_r := \min[\left(\partial h^{**}(\cdot)\right)^{-1}(m_r)] < \beta_r := \max[\left(\partial h^{**}(\cdot)\right)^{-1}(m_r)], \quad r = 1, 2, \ldots.
\]

Setting \( q_r := h^{**}(z_r) - m_z z_r \), one obtains

\[
h^{**}(\xi) = q_r + m_r \xi \quad \forall \xi \in [z_r, \beta_r].
\]

Clearly, \( h(z_r) = h^{**}(z_r) \) and \( h(\beta_r) = h^{**}(\beta_r) \). Moreover, the set

\[
K = \{\xi > 0 : h^{**}(\xi) < h(\xi)\}
\]

of positive noncontact points, of \( h(\cdot) \) with its bipolar \( h^{**}(\cdot) \), is contained in

\[
\bigcup_{r=1}^{+\infty} (z_r, \beta_r).
\]

By construction, the set

\[
D_+ := \{t \in [a,b] : \exists y'(t) > 0 \text{ and } h^{**}(y'(t)) < h(y'(t))\},
\]

satisfies the inclusion

\[
\left(\partial h^{**}(\cdot)\right)^{-1}(h^{**}(y'(D_+))) \subset (0, + \infty).
\]
(b) Define the measurable sets
\[ E_r := \{ t \in [a, b] : y'(t) \in (\alpha_r, \beta_r) \}, \]
\[ S_r := y(E_r) = \{ s \in [A, B] : \frac{1}{\nu_0(s)} \in (\alpha_r, \beta_r) \}. \]
\[ E = \bigcup_{r=1}^{\infty} E_r, \quad S := y(E), \quad E_0 := [a, b] \setminus E, \quad S_0 := y(E_0). \] (7)

Here \( \nu_0(s) := y^{-1}'(s) \) is the derivative of the inverse function \( y^{-1} : [A, B] \to [a, b] \) of the function \( y : [a, b] \to [A, B] \) at the point \( s \), so that
\[ s = y(t) \Leftrightarrow t = y^{-1}(s), \quad \nu_0(y(t)) = (y'(t))^{-1} \quad \text{a.e. on } [a, b]. \]

For any set \( Z \subset \mathbb{R} \), define its characteristic function \( \chi_Z(\cdot) \), equal to 1 on \( Z \) and to 0 on \( \mathbb{R} \setminus Z \). Notice that, by the change of variables formula in the Lebesgue integral (see e.g. [22, Corollary 6]), we have
\[
\int_{a}^{b} \varphi(y(t))\chi_{E_r}(t) \, dt = \int_{A}^{B} \varphi(s)\nu_0(s)\chi_{S_r}(s) \, ds,
\]
\[
\int_{a}^{b} \rho(y(t))h^{**}(y'(t))\chi_{E_r}(t) \, dt = \int_{A}^{B} \rho(s)h^{**}\left(\frac{1}{\nu_0(s)}\right)\nu_0(s)\chi_{S_r}(s) \, ds
\]
\[ = \int_{A}^{B} \rho(s)[q_r\nu_0(s) + m_r]\chi_{S_r}(s) \, ds. \]

Clearly, the value of the integral
\[ \int_{A}^{B} \rho(s)\chi_{S_r}(s) \, ds \]
is finite. The same happens with the integral
\[ \int_{A}^{B} \rho(s)\chi_{E_r}(s) \, ds \]
in case \( \rho(\cdot) \in L^1(S_r) \). Or in case \( r \) satisfies
\[ h^{**}(\xi) = q_r + m_r \xi > 0 \quad \forall \xi \in [\alpha_r, \beta_r]. \]

Or whenever \( q_r m_r = 0 \). Or even in case \( y'(t) \in \text{interior}[(\partial h^{**}(\cdot))^{-1}(\mathbb{R})] \) a.e., as is easily seen by applying the DuBois–Reymond differential inclusion (6) (see [3, Theorem 4.1]). Indeed, we then have \( \varphi(s) = c - q_r \rho(s) \) a.e. on \( S_r \), for some constant \( c \), in this case.

However, we have to face the fact that \( \rho(\cdot) \) may not belong to \( L^1(S_r) \), namely in case \( h^{**}(\xi) = 0 \) for either \( \xi = \alpha_r \) or \( \xi = \beta_r \) (with \( q_r m_r \neq 0 \)). Let us consider, say, the first case, in which we necessarily have \( h^{**}(\xi) = m_r(\xi - \alpha_r) \), for \( \xi \in [\alpha_r, \beta_r] \), with \( m_r > 0, \alpha_r > 0 \). In particular,
\[ \rho(s)h^{**}\left(\frac{1}{\nu_0(s)}\right)\nu_0(s)\chi_{S_r}(s) = \rho(s)m_r[1 - \alpha_r\nu_0(s)]\chi_{S_r}(s). \]
Define
\[ u_0(s) := \frac{1 - z_r v_0(s)}{\beta_r - z_r} \beta_r p(s). \]

Then \( u_0(s) \in (0, \rho(s)) \) a.e. on \( S_r \), and the values of the following integrals are finite:
\[ \int_A^B u_0(s) \chi_{S_r}(s) \, ds, \quad \int_A^B u_0(s) \frac{1}{\rho(s)} \chi_{S_r}(s) \, ds, \quad \int_A^B u_0(s) \frac{\varphi(s)}{\rho(s)} \chi_{S_r}(s) \, ds. \]

Define now, for each \( n \in \{0, 1, 2, \ldots\} \),
\[ \rho_n(s) := \min\{\rho(s), u_0(s) + n\} \]
and notice that \( 0 < u_0(s) \leq \rho_n(s) \leq u_0(s) + n \). In particular, setting \( w_n^0(s) := u_0(s)/\rho_n(s) \), we have \( w_n^0(\cdot) \in (0, 1) \) and
\[ \int_A^B u_0(s) \chi_{S_r}(s) \, ds = \int_A^B w_n^0(s) \rho_n(s) \chi_{S_r}(s) \, ds, \]
\[ \int_A^B u_0(s) \frac{1}{\rho(s)} \chi_{S_r}(s) \, ds = \int_A^B w_n^0(s) \frac{\rho_n(s)}{\rho(s)} \chi_{S_r}(s) \, ds, \]
\[ \int_A^B u_0(s) \frac{\varphi(s)}{\rho(s)} \chi_{S_r}(s) \, ds = \int_A^B w_n^0(s) \frac{\rho_n(s) \varphi(s)}{\rho(s)} \chi_{S_r}(s) \, ds, \]
with the functions \( \rho_n(\cdot), \rho_n(\cdot), \rho_n(\cdot) \varphi(\cdot)/\rho(\cdot) \in L^1(S_r) \).

Let us construct, by induction, a measurable function \( v_r : [A, B] \to \{1/\beta_r, 1/z_r\} \), satisfying adequate properties which will be needed in part (c) of the proof. Define \( u_n^0(\cdot) := u_0 \chi_{S_r}(\cdot) \), \( r = 1, 2, \ldots. \)

Before starting the induction step, we admit to have constructed a measurable function
\[ u_n^r : [A, B] \to [0, +\infty) \] with values
\[ u_n^r(s) \in \{0, \rho_n(s)\} \quad \text{a.e. on} \ S_r, \ u_n^r(s) = 0 \quad \text{a.e. on} \ [A, B] \setminus S_r, \]
satisfying
\[ \int_A^B u_n^r(s) \chi_{S_r}(s) \, ds = \int_A^B u_0(s) \chi_{S_r}(s) \, ds, \]
\[ \int_A^B u_n^r(s) \frac{1}{\rho(s)} \chi_{S_r}(s) \, ds = \int_A^B u_0(s) \frac{1}{\rho(s)} \chi_{S_r}(s) \, ds, \]
\[ \int_A^B u_n^r(s) \frac{\varphi(s)}{\rho(s)} \chi_{S_r}(s) \, ds = \int_A^B u_0(s) \frac{\varphi(s)}{\rho(s)} \chi_{S_r}(s) \, ds. \]
(8)

(This is certainly true for \( n = 0 \).)

In particular, with
\[ v_n^r(s) := \frac{1}{z_r} - \left( \frac{1}{z_r} - \frac{1}{\beta_r} \right) \frac{u_n^r(s)}{\rho(s)}, \]
we have
\[
\int_A^B u^n(s) \mathcal{Z}_S(s) \, ds = \int_A^B v_0(s) \mathcal{Z}_S(s) \, ds,
\]
\[
\int_A^B \varphi(s) u^n(s) \mathcal{Z}_S(s) \, ds = \int_A^B \varphi(s) v_0(s) \mathcal{Z}_S(s) \, ds,
\]
\[
\int_A^B \rho(s) m_r [1 - \alpha_r v^n(s)] \mathcal{Z}_S(s) \, ds = \int_A^B \rho(s) m_r [1 - \alpha_r v_0(s)] \mathcal{Z}_S(s) \, ds.
\]  
(9)

The induction step consists in constructing the new measurable function \( u^{n+1}(\cdot) \) starting from the knowledge of \( u^n(\cdot) \) and the set
\[
S^n_r := \{ s \in S_r : \rho_n(s) < \rho_{n+1}(s) \}.
\]

On its complement \([A, B] \setminus S^n_r\) define \( u^{n+1}(\cdot) := u^n(\cdot) \), obtaining
\[
\int_A^B u^{n+1}(s) \mathcal{Z}_S(s) \, ds = \int_A^B u^n(s) \mathcal{Z}_S(s) \, ds,
\]
\[
\int_A^B u^{n+1}(s) \frac{1}{\rho(s)} \mathcal{Z}_S(s) \, ds = \int_A^B u^n(s) \frac{1}{\rho(s)} \mathcal{Z}_S(s) \, ds,
\]
\[
\int_A^B u^{n+1}(s) \phi(s) \frac{1}{\rho(s)} \mathcal{Z}_S(s) \, ds = \int_A^B u^n(s) \phi(s) \frac{1}{\rho(s)} \mathcal{Z}_S(s) \, ds.
\]

Let us obtain now \( u^{n+1}(\cdot)|_{S^n_r} \). By the Liapunov Theorem on the range of vector measures (see e.g. \([4,8,15]\)), we can find a measurable function
\[
u^{n+1} : S^n_r \to [0, +\infty) \text{ with values}
\]
\[
u^{n+1}(s) \in \{0, \rho_{n+1}(s)\} \quad \text{a.e.,}
\]
satisfying
\[
\int_A^B u^{n+1}(s) \mathcal{Z}_{S^n}(s) \, ds = \int_A^B u^n(s) \mathcal{Z}_{S^n}(s) \, ds,
\]
\[
\int_A^B u^{n+1}(s) \frac{1}{\rho(s)} \mathcal{Z}_{S^n}(s) \, ds = \int_A^B u^n(s) \frac{1}{\rho(s)} \mathcal{Z}_{S^n}(s) \, ds,
\]
\[
\int_A^B u^{n+1}(s) \phi(s) \frac{1}{\rho(s)} \mathcal{Z}_{S^n}(s) \, ds = \int_A^B u^n(s) \phi(s) \frac{1}{\rho(s)} \mathcal{Z}_{S^n}(s) \, ds.
\]
Clearly, equalities (8) are then satisfied with \( n + 1 \) in place of \( n \). Since
\[
S^n_r = \{ s \in S_r : u_0(s) + n < \rho(s) \} \supset S^{n+1}_r
\]
and \( u_0(s), \rho(s) \in [0, +\infty) \) for a.e. \( s \) on \( S_r \), we have

\[
S_r = \bigcup_{n=1}^{+\infty} (S^n_r \setminus S^{n+1}_r).
\]

In particular, the measure of \( S^n_r \setminus S^{n+1}_r \) tends to zero as \( n \to +\infty \), and the sequence \( (u^n_r(\cdot)) \) converges, pointwise a.e. on \( S_r \), to a Lebesgue measurable function \( u_r(\cdot) \). Clearly \( u_r(s) \in \{0, \rho(s)\} \), a.e. on \( S_r \).

Given the affine dependence of \( v^n_r(\cdot) \) on \( u^n_r(\cdot) \), also the sequence \( (v^n_r(\cdot)) \) converges, pointwise a.e on \( S_r \), to the affine image \( v_r(\cdot) \) of \( u_r(\cdot) \), and we have \( v_r(s) \in \{1/\beta_r, 1/\gamma_r\} \)

By (9) and the dominated convergence theorem,

\[
\int_A^B v_r(s) \chi_S(s) \, ds = \int_A^B v_0(s) \chi_S(s) \, ds.
\]

Also by (9), together with Fatou’s lemma,

\[
\int_A^B v_r(s) \phi(s) \chi_S(s) \, ds \leq \int_A^B v_0(s) \phi(s) \chi_S(s) \, ds,
\]

\[
\int_A^B \rho(s) m_r [1 - \alpha_r v_r(s)] \chi_S(s) \, ds \leq \int_A^B \rho(s) m_r [1 - \alpha_r v_0(s)] \chi_S(s) \, ds.
\]

(c) This shows, in conclusion, that it is always possible to apply the Liapunov theorem and construct, for each \( r \in \{1, 2, \ldots\} \), a measurable function \( v_r : [A, B] \to (0, +\infty) \) such that

\[
\frac{1}{v_r(s)} \chi_S(s) \in \mathcal{C} \left[ (\partial h^{**}(\cdot))^{-1} \left( h^{**}(\frac{1}{v_0(s)}) \right) \right] \chi_S(s),
\]

\[
\int_A^B \chi_S(s) v_r(s) \, ds = \int_A^B \chi_S(s) v_0(s) \, ds,
\]

\[
\int_A^B \chi_S(s) v_r(s) \phi(s) \, ds = \int_A^B \chi_S(s) v_0(s) \phi(s) \, ds,
\]

\[
\int_A^B \chi_S(s) v_r(s) \rho(s) h \left( \frac{1}{v_r(s)} \right) \, ds \leq \int_A^B \chi_S(s) v_0(s) \rho(s) h^{**} \left( \frac{1}{v_0(s)} \right) \, ds.
\]

Define a new function

\[
\tau : [A, B] \to [a, +\infty],
\]

\[
\tau(s) := a + \int_A^s \sum_{r=0}^{+\infty} \chi_S(\sigma) v_r(\sigma) \, d\sigma.
\]
Notice that $\tau(A) = a$ and, by Wheeden and Zygmund [23, Theorem 5.24],

$$
\tau(B) = a + \int_A^B \sum_{r=0}^{+\infty} \chi_S(s)v_r(s) \, ds
$$

$$
= a + \sum_{r=0}^{+\infty} \int_A^B \chi_S(s)v_r(s) \, ds = a + \sum_{r=0}^{+\infty} \int_A^B \chi_S(s)v_0(s) \, ds
$$

$$
= a + \int_A^B \left( \sum_{r=0}^{+\infty} \chi_S(s) \right) v_0(s) \, ds = a + \int_A^B y^{-1}(s) \, ds = b.
$$

This shows that $\tau: [A, B] \to [a, b]$ is absolutely continuous and has derivative

$$
\tau'(s) = \sum_{r=0}^{+\infty} \chi_S(s)v_r(s) > 0 \quad \text{a.e. on } [A, B],
$$

in particular, $\tau(\cdot)$ increases strictly onto $[a, b]$, hence its inverse function $x: [a, b] \to [A, B]$ is well-defined, absolutely continuous and strictly increasing, with $x(a) = A$, $x(b) = B$ and derivative

$$
x'(t) = (\tau'(x(t)))^{-1} \quad \text{a.e. on } [a, b]. \quad (10)
$$

Since $t = \tau(s) \iff s = x(t),

$$
\int_a^b \{ \rho(x(t)) h^{**}(x'(t)) + \varphi(x(t)) \} \, dt
$$

$$
\leq \int_a^b \{ \rho(x(t)) h'(x'(t)) + \varphi(x(t)) \} \, dt = \int_A^B \left\{ \rho(s) h \left( \frac{1}{\tau(s)} \right) + \varphi(s) \right\} \tau'(s) \, ds
$$

$$
= \int_A^B \sum_{r=0}^{+\infty} \chi_S(s)v_r(s)\rho(s) \left( \frac{1}{v_r(s)} \right) \, ds + \int_A^B \sum_{r=0}^{+\infty} \chi_S(s)v_r(s)\varphi(s) \, ds
$$

$$
= \sum_{r=0}^{+\infty} \int_A^B \chi_S(s)v_r(s)\rho(s) \left( \frac{1}{v_r(s)} \right) \, ds + \sum_{r=0}^{+\infty} \int_A^B \chi_S(s)v_r(s)\varphi(s) \, ds
$$

$$
\leq \sum_{r=0}^{+\infty} \int_A^B \chi_S(s)v_0(s)\rho(s) h^{**} \left( \frac{1}{v_0(s)} \right) \, ds + \sum_{r=0}^{+\infty} \int_A^B \chi_S(s)v_0(s)\varphi(s) \, ds
$$

$$
= \int_A^B \left\{ \rho(s) h^{**} \left( \frac{1}{v_0(s)} \right) + \varphi(s) \right\} v_0(s) \, ds
$$

$$
= \int_a^b \{ \rho(y(t)) h^{**}(y'(t)) + \varphi(y(t)) \} \, dt,
$$

again by the formula for change of variables.

(d) We have thus obtained a minimizer $x(\cdot)$ of the nonconvex integral (4). Clearly the regularity properties (i) and (ii) are inherited, by $x(\cdot)$, from $y(\cdot)$. As to (iv) and (v), they have been proved in [19, Theorem 1, (iv) and (v)].
Finally, let us prove property (iii). Clearly if \( y'(t) \in \text{interior } [(h^{**}(\cdot))^{-1}(\mathbb{R})] \) a.e.,
then \( y(\cdot) \) satisfies the DuBois–Reymond differential inclusion (see [3, Theorem 4.1]):
there exists a constant \( c \) such that, defining \( q(s) := [c - \varphi(s)]/\rho(s) \),
\[
h^{**}(y'(t)) \in q(y(t)) + y'(t) \partial h^{**}(y(t)) \quad \text{a.e. on } [a,b].
\] (11)

To show the validity of this inclusion also for \( x(\cdot) \), assume again, without loss of
generality, \( y(\cdot) \) to increase strictly on \([a,b], \text{ onto } [A,B]. \) With \( v_0(s) := y^{-1}(s), \) and
\[
t = y^{-1}(s) \in [a,b] \iff s = y(t) \in [A,B], \quad y'(t) = 1/v_0(s)
\]
we have
\[
q(s) = q(y(t)) \in h^{**}(y'(t)) - y'(t) \partial h^{**}(y(t))
\]
\[
= h^{**} \left( \frac{1}{v_0(s)} \right) - \frac{1}{v_0(s)} \partial h^{**} \left( \frac{1}{v_0(s)} \right),
\]
for a.e. \( t \in [a,b] \) and a.e. \( s \) in \([A,B]. \) Therefore, with \( \tau(s) := x^{-1}(s), \) we have
\[
s = x(t) \in [A,B] \iff t = \tau(s) \in [a,b], \quad x'(t) = 1/\tau'(s)
\]

hence, taking into consideration the construction of \( \tau'(\cdot), \)
\[
q(x(t)) = q(s) \in h^{**} \left( \frac{1}{\tau'(s)} \right) - \frac{1}{\tau'(s)} \partial h^{**} \left( \frac{1}{\tau'(s)} \right)
\]
\[
\subset h \left( \frac{1}{\tau'(s)} \right) - \frac{1}{\tau'(s)} \partial h^{**} \left( \frac{1}{\tau'(s)} \right) = h(x'(t)) - x'(t) \partial h^{**}(x'(t)),
\]
for a.e. \( s \) in \([A,B] \) and a.e. \( t \) in \([a,b]. \) Therefore (11) holds true with \( x(\cdot) \) in place of
\( y(\cdot), \) i.e. (6) holds true.

**Remark 2.** We imposed, in the theorem, \( \varphi(\cdot) \) bounded below just for simplicity.
Actually, as is usual in the application of the direct method, it suffices to ask for
\[
\varphi(s) \geq - \gamma_1 - \gamma_2|s|^p, \quad p \geq 1,
\]
provided a stronger growth condition is imposed on \( h(\cdot) \) in case \( p > 1: \)
\[
h(\xi) \geq - \gamma_3 + \gamma_4|\xi|^q,
\]
with either \( q > p \) and \( \gamma_4 = 1 \) or \( q = p \) and \( \gamma_4 > 0 \) large enough relative to \( b - a \)
and \( \gamma_2 > 0 \) (see e.g. [8–10]). Also \( h(\cdot) \) is supposedly \( \geq 0 \) for the same reason; otherwise
define \( h_+(\xi) := h(\xi) - \min h(\mathbb{R}), \)
\[
\varphi_+(s) := \varphi(s) + \rho(s) \min h(\mathbb{R}), \quad \text{so that } h_+(\cdot) \geq 0
\]
and \( \rho(s)h_+(\xi) + \varphi(s) = \rho(s)h_+(\xi) + \varphi_+(s), \) and assume \( \varphi_+(\cdot) \) lower semicontinuous and
bounded below.

**References**

[5] G. Aubert, R. Tahraoui, Théoremes d’existence pour des problèmes du calcul des variations du type: Inf \( \int_0^1 f(x, u'(x)) \, dx \) et Inf \( \int_0^1 f(x, u(x), u'(x)) \, dx \), J. Differential Equations 33 (1979) 1–15.