On some third order nonlinear boundary value problems: Existence, location and multiplicity results

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Abstract

We prove an Ambrosetti–Prodi type result for the third order fully nonlinear equation

\[ u'''(t) + f(t, u(t), u'(t), u''(t)) = sp(t) \]

with \( f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R} \) and \( p : [0, 1] \to \mathbb{R}^+ \) continuous functions, \( s \in \mathbb{R} \), under several two-point separated boundary conditions. From a Nagumo-type growth condition, an \textit{a priori} estimate on \( u'' \) is obtained. An existence and location result will be proved, by degree theory, for \( s \in \mathbb{R} \) such that there exist lower and upper solutions. The location part can be used to prove the existence of positive solutions if a non-negative lower solution is considered. The existence, nonexistence and multiplicity of solutions will be discussed as \( s \) varies.

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1. Introduction

In this paper we study the following third order fully nonlinear equation

\[ u'''(t) + f(t, u(t), u'(t), u''(t)) = sp(t), \quad (E_s) \]

for \( f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R} \) and \( p : [0, 1] \to \mathbb{R}^+ \) continuous functions and \( s \) a real parameter, with several types of two-point boundary conditions.

If the boundary conditions are

\[ u(0) = A, \quad au'(0) - bu''(0) = B, \quad cu'(1) + du''(1) = C, \]

(1)

for \( a, b, c, d, A, B, C \in \mathbb{R} \) and \( b, d \geq 0 \) such that \( a^2 + b > 0 \) and \( c^2 + d > 0 \) an existence result is proved, for values of \( s \) such that there are lower and upper solutions to the problem \((E_s)\)–(1).
In Section 3 we consider boundary conditions
\[ u(0) = 0, \quad au'(0) - bu''(0) = 0, \quad cu'(1) + du''(1) = 0 \] (2)
with \( a, b, c, d \geq 0 \) such that \( a + b > 0, \ c + d > 0 \) and proving that the existence of solutions for the problem (E_s)--(2) depends on \( s \).

Considering, in (2), \( b = d = 0 \) with \( a, c > 0 \) the two-point boundary conditions are
\[ u(0) = u'(0) = u'(1) = 0, \] (3)
an Ambrosetti–Prodi type result is obtained in Section 4. That is, we prove that there are \( s_0, s_1 \in \mathbb{R} \) such that (E_s)--(3) has no solution if \( s < s_0 \), it has at least one solution if \( s = s_0 \) and (E_s)--(3) has at least two solutions for \( s \in [s_0, s_1] \).

Equation (E_s) can be seen as a generalized model for various physical, natural or physiological phenomena such as the flow of a thin film of viscous fluid over a solid surface [1,12], the solitary waves solution of the Korteweg–de Vries equation [8] or the thyroid-pituitary interaction [3]. The problem (E_s)--(1) can model the static deflection of an elastic beam with linear supports at both endpoints.

The arguments used were suggested by several papers namely [4], applied to second order periodic problems [11], to third order three points boundary value problems [5–7], for two-point boundary value problems. In short, they make use of a Nagumo-type growth condition [10], the upper and lower solutions technique [2], and Leray–Schauder degree theory [9].

2. Preliminary results

In the following, \( C([0, 1]) \) denotes the space of continuous functions with the norm
\[ \|x\| = \max_{t \in [0, 1]} |x(t)|. \]
Moreover, \( C^k([0, 1]) \) denotes the space of real valued functions with continuous \( i \)-derivative in \([0, 1]\), for \( i = 1, \ldots, k \), equipped with the norm
\[ \|x\|_{C^k} = \max_{0 \leq i \leq k} \{ |x^{(i)}(t)| : t \in [0, 1] \}. \]

Some growth conditions on the nonlinearity of (E_s) will be assumed in the following. The first one is given by the next definition and provides also an \textit{a priori} estimate for the second derivative of solutions \( u \) of (E_s), if some bounds on \( u \) and \( u' \) are verified.

\textbf{Definition 1.} A continuous function \( g : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R} \) is said to satisfy Nagumo-type condition in
\[ E = \{(t, x, y, z) \in [0, 1] \times \mathbb{R}^3 : \gamma_0(t) \leq x \leq \Gamma_0(t), \ \gamma_1(t) \leq y \leq \Gamma_1(t)\}, \]
with \( \gamma_0, \gamma_1, \Gamma_0 \) and \( \Gamma_1 \) continuous functions such that \( \gamma_0(t) \leq \Gamma_0(t), \ \gamma_1(t) \leq \Gamma_1(t) \), for every \( t \in [0, 1] \), if there exists a continuous function \( h_E : \mathbb{R}_0^+ \rightarrow [k, +\infty] \), for some fixed \( k > 0 \), such that
\[ |g(t, x, y, z)| \leq h_E(|z|), \quad \forall (t, x, y, z) \in E, \] (4)
with
\[ \int_0^{+\infty} \frac{\xi}{h_E(\xi)} d\xi = +\infty. \] (5)

If these assumptions hold for every \( E \subset [0, 1] \times \mathbb{R}^3 \), given above, then \( g \) is said to satisfy Nagumo-type conditions.

\textbf{Lemma 2.} Let \( f : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R} \) be a continuous function that satisfies Nagumo-type conditions (4) and (5) in
\[ E = \{(t, x, y, z) \in [0, 1] \times \mathbb{R}^3 : \gamma_0(t) \leq x \leq \Gamma_0(t), \ \gamma_1(t) \leq y \leq \Gamma_1(t)\}, \] (6)
where \( \gamma_0, \gamma_1, \Gamma_0, \Gamma_1 \) are continuous functions. Then there is \( r_s > 0 \) (depending only on the parameter \( s \) and on the functions \( p, h_E, \gamma_1 \) and \( \Gamma_1 \)) such that every solution \( u(t) \) of (E_s) verifying

\[ \gamma_0(t) \leq u(t) \leq \Gamma_0(t), \quad \gamma_1(t) \leq u'(t) \leq \Gamma_1(t) \]

for every \( t \in [0,1] \), satisfies

\[ \|u''\| < r^*. \]

**Remark 1.** We observe that \( r^* \) can be taken independent of \( s \) as long as \( s \) belongs to some bounded set.

**Proof.** Considering the non-negative number

\[ \eta = \max \{ \Gamma_1(1) - \gamma_1(0), \Gamma_1(0) - \gamma_1(1) \} \]

and \( r > \eta \) such that

\[ \int_{\eta}^{r} \frac{\xi}{h_E(\xi) + |s||p|} \, d\xi \geq \max_{t \in [0,1]} \Gamma_1(t) - \min_{t \in [0,1]} \gamma_1(t), \]

then the proof follows from [5, Lemma 1], as \((E_s)\) is a particular case of the equation there assumed. \( \Box \)

The appropriate definition of lower and upper-solutions for problem \((E_s)-(1)\) is now given.

**Definition 3.** Consider \( a, b, c, d, A, B, C \in \mathbb{R} \) such that \( b, d \geq 0, a^2 + b > 0 \) and \( c^2 + d > 0 \).

(i) A function \( \alpha(t) \in C^3([0,1]) \cap C^2([0,1]) \) is a lower solution of \((E_s)-(1)\) if

\[ \alpha'''(t) + f(t, \alpha(t), \alpha'(t), \alpha''(t)) \geq sp(t), \quad \text{if } t \in ]0,1[, \]

and

\[ \alpha(0) \leq A, \quad a\alpha'(0) - ba''(0) \leq B, \quad c\alpha'(1) + da''(1) \leq C. \]

(ii) A function \( \beta(t) \in C^3([0,1]) \cap C^2([0,1]) \) is an upper solution of \((E_s)-(1)\) if

\[ \beta'''(t) + f(t, \beta(t), \beta'(t), \beta''(t)) \leq sp(t), \quad \text{if } t \in ]0,1[, \]

and

\[ \beta(0) \geq A, \quad a\beta'(0) - b\beta''(0) \geq B, \quad c\beta'(1) + d\beta''(1) \geq C. \]

For \( s \) such that there are upper and lower solutions of \((E_s)-(1)\) with first derivative “well ordered,” an existence result and some information concerning the location of the solution of \((E_s)-(1)\) and its derivative are obtained.

**Theorem 4.** Let \( f : [0,1] \times \mathbb{R}^3 \to \mathbb{R} \) be a continuous function. Suppose that there are lower and upper solutions of \((E_s)-(1)\), \( \alpha(t) \) and \( \beta(t) \), respectively, such that, for \( t \in [0,1] \),

\[ \alpha'(t) \leq \beta'(t) \]

and \( f \) satisfies Nagumo-type conditions (4) and (5) in

\[ E_s = \{(t, x, y, z) \in [0,1] \times \mathbb{R}^3 : \alpha(t) \leq x \leq \beta(t), \alpha'(t) \leq y \leq \beta'(t) \}. \]

If \( f \) verifies

\[ f(t, \alpha(t), y, z) \leq f(t, x, y, z) \leq f(t, \beta(t), y, z), \]

for fixed \( t, y, z \in [0,1] \times \mathbb{R}^2 \) and \( \alpha(t) \leq x \leq \beta(t) \), then \((E_s)-(1)\) has at least one solution \( u(t) \in C^3([0,1]) \) satisfying

\[ \alpha(t) \leq u(t) \leq \beta(t), \quad \alpha'(t) \leq u'(t) \leq \beta'(t), \quad \forall t \in [0,1]. \]
Proof. Define the auxiliary continuous functions

\[
\delta_0(t, x) = \begin{cases} 
\beta(t) & \text{if } x > \beta(t), \\
\alpha(t) & \text{if } x < \alpha(t), \\
x & \text{if } \alpha(t) \leq x \leq \beta(t),
\end{cases}
\]

and

\[
\delta_1(t, y) = \begin{cases} 
\beta'(t) & \text{if } y > \beta'(t), \\
\alpha'(t) & \text{if } y < \alpha'(t), \\
y & \text{if } \alpha'(t) \leq y \leq \beta'(t),
\end{cases}
\]

and, for \(\lambda \in [0, 1]\), the modified problem composed, by

\[
u''(t) + \lambda f(t, \delta_0(t, u(t)), \delta_1(t, u'(t)), u''(t)) - u'(t) + \lambda \delta_1(t, u'(t)) = \lambda sp(t)
\]

and the boundary conditions

\[
\begin{align*}
u(0) &= \lambda A, \\
u'(0) &= \lambda \left[ B - a \delta_1(0, u'(0)) + bu''(0) + \delta_1(0, u'(0)) \right], \\
u'(1) &= \lambda \left[ C - c \delta_1(1, u'(1)) - du''(1) + \delta_1(1, u'(1)) \right].
\end{align*}
\]

Taking \(r_1 > 0\) such that, for every \(t \in [0, 1]\),

\[
\begin{align*}
-r_1 &\leq \alpha'(t) \leq \beta'(t) \leq r_1, \\
sp(t) - f(t, \alpha(t), \alpha'(t), 0) - r_1 - \alpha'(t) &< 0, \\
sp(t) - f(t, \beta(t), \beta'(t), 0) + r_1 - \beta'(t) &> 0
\end{align*}
\]

and

\[
\begin{align*}
|B + (1-a)\beta'(0)| &< r_1, \\
|B + (1-a)\alpha'(0)| &< r_1, \\
|C + (1-c)\beta'(1)| &< r_1, \\
|C + (1-c)\alpha'(1)| &< r_1
\end{align*}
\]

the proof follows the arguments used in [5, Theorem 1]. So, only the following details due to a more general boundary conditions are included.

In Step 1 it is proved that every solution \(u\) of (10)–(11) satisfies \(|u'(t)| < r_1\) and \(|u(t)| < r_0\), for every \(t \in [0, 1]\) and \(r_0 := r_1 + |A|\), independently of \(\lambda\).

In Step 2, the set

\[E_r = \{ (t, x, y, z) \in [0, 1] \times \mathbb{R}^3; |x| \leq r_0, |y| \leq r_1 \}\]

and the function \(F_\lambda : E_r \rightarrow \mathbb{R}\) given by

\[
F_\lambda(t, x, y, z) := \lambda f(t, \delta_0(t, x), \delta_1(t, y), z) - y + \lambda \delta_1(t, y)
\]

are considered. As \(|F_\lambda(t, x, y, z)| \leq 2r_1 + h_{E_\lambda}(|z|)\) and

\[
\int_0^{+\infty} \frac{z}{2r_1 + h_{E_\lambda}(z)} \, dz = +\infty
\]

then \(F_\lambda\) satisfies a Nagumo-type condition in \(E_\lambda\) and the assumptions of Lemma 2 are verified.

In Step 3 the nonlinear operator \(N_\lambda\) is defined by

\[
N_\lambda u = (-\lambda f(t, \delta_0(t, u(t)), \delta_1(t, u'(t)), u''(t)) + u'(t) - \lambda \delta_1(t, u'(t)) + \lambda sp(t), \lambda A, B_\lambda, C_\lambda)
\]

with

\[
B_\lambda := \lambda \left[ B - a \delta_1(0, u'(0)) + bu''(0) + \delta_1(0, u'(0)) \right], \\
C_\lambda := \lambda \left[ C - c \delta_1(1, u'(1)) - du''(1) + \delta_1(1, u'(1)) \right]
\]

and the Leray–Schauder degree is evaluated in the set

\[\Omega = \{ x \in C^2([0, 1]): \|x\| < r_0, \|x'\| < r_1, \|x''\| < r_2 \} \]
Example. Consider the differential equation

\[ u''(t) + |u''(t)|^\theta - k[u'(t)]^{2n+1} + [u(t)]^{2m+1} = sp(t) \tag{12} \]

for \( t \in [0, 1], \theta \in [0, 2], n, m \in \mathbb{N}, k > 0, s \in \mathbb{R} \) and \( p : [0, 1] \to \mathbb{R}^+ \) a continuous function, with the boundary conditions

\[ u(0) = 0, \quad au'(0) - bu''(0) = B, \quad cu'(1) + du''(1) = C, \tag{13} \]

for \( B, C \in \mathbb{R}, a, b, c, d \geq 0 \) with \( a + b > 0 \) and \( c + d > 0 \).

If \( a, c, B \) and \( C \) are such that \( |B| \leq a \) and \( |C| \leq c \) then functions \( \alpha, \beta : [0, 1] \to \mathbb{R} \) given by \( \alpha(t) = -t \) and \( \beta(t) = t \)

are, respectively, lower and upper solutions of problem (12)--(13) for \( |s| \leq \frac{k}{\|p\|} \). As

\[ f(t, x, y, z) = |z|^\theta - ky^{2n+1} + x^{2m+1} \]

is continuous and verifies Nagumo-type assumptions (4) and (5) in

\[ E = \{(t, x, y, z) \in [0, 1] \times \mathbb{R}^3 : |x| \leq t, \ |y| \leq 1\} \tag{14} \]

for \( h_E(z) = k + 1 + |z|^\theta \) then, by Theorem 4, problem (12) has at least one solution \( u(t) \) such that

\[ -t \leq u(t) \leq t, \quad -1 \leq u'(t) \leq 1, \quad \forall t \in [0, 1], \]

for \( |s| \leq \frac{k}{\|p\|} \).

3. Existence and nonexistence results

A first discussion concerning the dependence on \( s \) of the existence and nonexistence of a solution will be given in the special case that \( A = B = C = 0 \) and \( a, b, c, d \geq 0 \) with \( a + b > 0, c + d > 0 \), that is, for \((E_s)-(2)\). Lower and upper solutions definitions for this problem are obtained considering in Definition 3 these restrictions.

**Theorem 5.** Let \( f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R} \) be a continuous function satisfying a Nagumo-type condition and such that

(i) for \((t, y, z) \in [0, 1] \times \mathbb{R}^2\)

\[ x_1 \geq x_2 \Rightarrow f(t, x_1, y, z) \geq f(t, x_2, y, z); \tag{15} \]

(ii) for \((t, x, z) \in [0, 1] \times \mathbb{R}^2\)

\[ y_1 \geq y_2 \Rightarrow f(t, x, y_1, z) \leq f(t, x, y_2, z); \tag{16} \]

(iii) there are \( s_1 \in \mathbb{R} \) and \( r > 0 \) such that

\[ \frac{f(t, 0, 0, 0)}{p(t)} < s_1 < \frac{f(t, x, -r, 0)}{p(t)}, \tag{17} \]

for every \( t \in [0, 1] \) and every \( x \leq -r \). Then there is \( s_0 \leq s_1 \) (with the possibility that \( s_0 = -\infty \)) such that

(1) for \( s < s_0 \), \((E_s)-(2)\) has no solution;

(2) for \( s_0 < s \leq s_1 \), \((E_s)-(2)\) has at least one solution.

**Proof.** Step 1. There is \( s^* < s_1 \) such that \((E_{s^*})-(2)\) has a solution.

Defining

\[ s^* = \max \left\{ \frac{f(t, 0, 0, 0)}{p(t)}, \ t \in [0, 1] \right\}, \]

by (17), there exists \( t^* \in [0, 1] \) such that

\[ \frac{f(t, 0, 0, 0)}{p(t)} \leq s^* = \frac{f(t^*, 0, 0, 0)}{p(t^*)} < s_1, \quad \forall t \in [0, 1], \]

and, by the first inequality, \( \beta(t) \equiv 0 \) is an upper solution of \((E_{s^*})-(2)\).
The function $\alpha(t) = -rt$ is a lower solution of $(E_x^*)-(2)$. In fact, as $\alpha(t) \geq -r$, $\alpha'(t) = -r$ and $\alpha''(t) = \alpha'''(t) = 0$, then, by (17) and (15),

$$\alpha'''(t) = 0 > s_1 p(t) - f(t, -r, -r, 0) \geq s_1 p(t) - f(t, -rt, -r, 0) > s^* p(t) - f(t, -rt, -r, 0).$$

(18)

So, by Theorem 4, there is, at least a solution of $(E_x^*)-(2)$ with $s^* < s_1$.

Step 2. If $(E_x)-(2)$ has a solution for $s = \sigma < s_1$, then it has at least one solution for $s \in [\sigma, s_1]$.

Suppose that $(E_\sigma)-(2)$ has a solution $u_\sigma(t)$. For $s$ such that $\sigma \leq s \leq s_1$,

$$u_\sigma''(t) = \sigma p(t) - f(t, u_\sigma(t), u_\sigma'(t), u_\sigma''(t)) \leq p(t) - f(t, u_\sigma(t), u_\sigma'(t), u_\sigma''(t))$$

and so $u_\sigma(t)$ is an upper solution of $(E_s)-(2)$ for every $s$ such that $\sigma \leq s \leq s_1$.

For $r > 0$ given by (17) take $R \geq r$ large enough such that

$$u_\sigma'(0) \geq -R, \quad u_\sigma'(1) \geq -R \quad \text{and} \quad \min_{t \in [0,1]} u_\sigma(t) \geq -R.$$ 

(19)

Since, by (17) and (15), for $s \leq s_1$,

$$0 > s_1 p(t) - f(t, -R, -r, 0) \geq s p(t) - f(t, -Rt, -R, 0)$$

and $-aR \leq 0, -cR \leq 0$ then $\alpha(t) = -Rt$ is a lower solution of $(E_s)-(2)$ for $s \leq s_1$.

To apply Theorem 4 the condition

$$-R \leq u_\sigma'(t), \quad \forall t \in [0,1],$$

(20)

must be verified. Suppose that (20) is not true. Therefore there is $t \in [0,1]$ such that $u_\sigma'(t) < -R$. Defining

$$\min_{t \in [0,1]} u_\sigma'(t) := u_\sigma'(t_0) \quad (< -R)$$

then, by (19), $t_0 \in ]0, 1[, u_\sigma''(t_0) = 0$, $u_\sigma''(t_0) \geq 0$ and, by (16), (19) and (17), the following contradiction

$$0 \leq u_\sigma''(t_0) = \sigma p(t_0) - f(t_0, u_\sigma(t_0), u_\sigma'(t_0), u_\sigma''(t_0)) \leq \sigma p(t_0) - f(t_0, u_\sigma(t_0), -R, 0) \leq s_1 p(t_0) - f(t_0, -R, -R, 0) < 0$$

is obtained. So $-R \leq u_\sigma'(t)$, for every $t \in [0,1]$, and, by Theorem 4, problem $(E_x)-(2)$ has at least a solution $u(t)$ for every $s$ such that $\sigma \leq s \leq s_1$.

Step 3. There is $s_0 \in \mathbb{R}$ such that:

- for $s < s_0$, $(E_x)-(2)$ has no solution;
- for $s \in [s_0, s_1]$, $(E_x)-(2)$ has at least a solution.

Let $S = \{s \in \mathbb{R} : (E_s)-(2)$ has at least a solution$\}$. As, by Step 1, $s^* \in S$ then $S \neq \emptyset$. Defining $s_0 = \inf S$, by Step 1, $s_0 \leq s^* < s_1$ and, by Step 2, $(E_x)-(2)$ has at least a solution for $s \in [s_0, s_1]$ and $(E_x)-(2)$ has no solution for $s < s_0$.

Observe that if $s_0 = -\infty$ then, by Step 2, $(E_x)-(2)$ has a solution for every $s \leq s_1$.  

A variant of Theorem 5 can be obtained replacing, in (17), $f$ by $-f$ and $x$ by $-x$.

**Theorem 6.** Let $f : [0,1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be a continuous function satisfying a Nagumo-type condition and growth assumptions (15) and (16). If there are $s_1 \in \mathbb{R}$ and $r > 0$ such that

$$\frac{f(t, 0, 0, 0)}{p(t)} > s_1 > \frac{f(t, x, r, 0)}{p(t)},$$

for every $t \in [0,1]$ and every $x \geq r$, then there is $s_0 > s_1$ (with the possibility that $s_0 = +\infty$) such that

1. for $s > s_0$, $(E_x)-(2)$ has no solution;
2. for $s_0 > s \geq s_1$, $(E_x)-(2)$ has at least one solution.
4. Multiplicity results

In the particular case of boundary conditions (1) where \( b = d = A = B = C = 0 \) and \( a, c > 0 \) is proved the existence of a second solution for problem \((E_s)–(3)\) as a consequence of a non-null degree for the same operator in two disjoint sets.

The arguments are based on strict lower and upper solutions and some new assumptions on the nonlinearity.

**Definition 7.** Consider \( \alpha, \beta : [0, 1] \to \mathbb{R} \) such that \( \alpha, \beta \in C^3([0, 1]) \cap C^2([0, 1]) \).

(i) \( \alpha(t) \) is a strict lower solution of \((E_s)–(3)\) if

\[
\alpha''(t) + f(t, \alpha(t), \alpha'(t), \alpha''(t)) > sp(t), \quad \text{if } t \in ]0, 1[,
\]

and

\[
\alpha(0) \leq 0, \quad \alpha'(0) < 0, \quad \alpha'(1) < 0. \tag{21}
\]

(ii) \( \beta(t) \) is a strict upper solution of \((E_s)–(3)\) if

\[
\beta''(t) + f(t, \beta(t), \beta'(t), \beta''(t)) < sp(t), \quad \text{if } t \in ]0, 1[,
\]

and

\[
\beta(0) \geq 0, \quad \beta'(0) > 0, \quad \beta'(1) > 0. \tag{22}
\]

Define the set \( X = \{ x \in C^2([0, 1]) : x(0) = x'(0) = x'(1) = 0 \} \) and the operators \( L : \text{dom}L \to C([0, 1]), \) with \( \text{dom}L = C^3([0, 1]) \cap X \), given by \( Lu = u''' \) and, for \( s \in \mathbb{R}, \ N_s : C^2([0, 1]) \cap X \to C([0, 1]) \) given by

\[
N_su = f(t, u(t), u'(t), u''(t)) - sp(t).
\]

For an open and bounded set \( \Omega \subset X \), the operator \( L + N_s \) is \( L \)-compact in \( \overline{\Omega} \) [9]. Note that in \( \text{dom}L \) the equation \( Lu + N_su = 0 \) is equivalent to problem \((E_s)–(3)\).

The next result will be an important tool used to evaluate the Leray–Schauder topological degree.

**Lemma 8.** Consider a continuous function \( f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R} \) verifying a Nagumo-type condition and (15). If there are strict lower and upper solutions of \((E_s)–(3)\), \( \alpha(t) \) and \( \beta(t) \), respectively, such that

\[
\alpha'(t) < \beta'(t), \quad \forall t \in [0, 1],
\]

then there is \( \rho_2 > 0 \) such that \( d(L + N_s, \Omega) = \pm 1 \) for

\[
\Omega = \{ x \in \text{dom}L : \alpha(t) < x(t) < \beta(t), \ \alpha'(t) < x'(t) < \beta'(t), \ \| x'' \| < \rho_2 \}.
\]

**Remark 2.** The set \( \Omega \) can be taken the same for \((E_s)–(3)\), independent of \( s \), as long as \( \alpha \) and \( \beta \) are strict lower and upper solutions for \((E_s)–(3)\) and \( s \) belongs to a bounded set.

**Proof.** For the auxiliary functions \( \delta_0, \delta_1 \) defined in (8) and (9) consider the modified problem

\[
\begin{cases}
\begin{align*}
&u'''(t) + F(t, u(t), u'(t), u''(t)) = sp(t), \\
&u(0) = u'(0) = u'(1) = 0,
\end{align*}
\end{cases} \tag{23}
\]

where \( F : [0, 1] \times \mathbb{R}^3 \to \mathbb{R} \) is the continuous function given by

\[
F(t, x, y, z) = f(t, \delta_0(t, x), \delta_1(t, y), z) - y + \delta_1(t, y)
\]

and define the operator \( F_s : C^2([0, 1]) \cap X \to C([0, 1]) \) by

\[
F_su = F(t, u(t), u'(t), u''(t)) - sp(t).
\]
With these definitions problem (23) is equivalent to the equation $Lu + F_su = 0$ in $\text{dom} L$. For $\lambda \in [0, 1]$ and $u \in \text{dom} L$ consider the homotopy

$$H_\lambda u := Lu - (1 - \lambda)u'' + \lambda F_\lambda u$$

and take $\rho_1 > 0$ large enough such that, for every $t \in [0, 1]$,

$$-\rho_1 \leq \alpha'(t) < \beta'(t) \leq \rho_1,$$

$$sp(t) - f(t, \alpha(t), \alpha'(t), 0) - \rho_1 - \alpha'(t) < 0$$

and

$$sp(t) - f(t, \beta(t), \beta'(t), 0) + \rho_1 - \beta'(t) > 0.$$ 

Following the arguments referred in the proof of Theorem 4, there is $\rho_2 > 0$ such that every solution $u(t)$ of $H_\lambda u = 0$ satisfies $\|u'\| < \rho_1$ and $\|u''\| < \rho_2$, independently of $\lambda \in [0, 1]$. Defining

$$\Omega_1 = \{ x \in \text{dom} L: \|x'\| < \rho_1, \|x''\| < \rho_2 \}$$

then, every solution $u$ of $H_\lambda u = 0$ belongs to $\Omega_1$ for every $\lambda \in [0, 1], u \notin \partial \Omega_1$ and the degree $d(H_\lambda, \Omega_1)$ is well defined, for every $\lambda \in [0, 1]$.

For $\lambda = 0$ the equation $H_0 u = 0$, that is, the linear problem

$$\begin{cases} u'''(t) - u''(t) = 0, \\ u(0) = u'(0) = u''(1) = 0 \end{cases}$$

has only the trivial solution and, by degree theory, $d(H_0, \Omega_1) = \pm 1$. By the invariance under homotopy

$$\pm 1 = d(H_0, \Omega_1) = d(H_1, \Omega_1) = d(L + F_\lambda, \Omega_1). \quad (24)$$

In the sequel it is proved that if $u \in \Omega_1$ is a solution of $Lu + F_\lambda u = 0$ then $u \in \Omega$.

In fact, by (24), there is $u_1(t) \in \Omega_1$ solution of $Lu + F_\lambda u = 0$. Assume, by contradiction, that there is $t \in [0, 1]$ such that $u_1(t) \leq \alpha'(t)$ and define

$$\min_{t \in [0, 1]} [u_1(t) - \alpha'(t)] := u_1'(t_1) - \alpha'(t_1) \quad (\leq 0).$$

From (21) $t_1 \in [0, 1], u_1''(t_1) - \alpha''(t_1) = 0$ and $u_1'''(t_1) - \alpha'''(t_1) \geq 0$. By (15), the following contradiction:

$$u_1'''(t_1) = sp(t_1) - F(t_1, u_1(t_1), u_1'(t_1), u_1''(t_1))$$

$$= sp(t_1) - f(t_1, \delta_0(t_1, u_1(t_1)), \delta_1(t_1, u_1'(t_1)), u_1''(t_1)) + u_1'(t_1) - \delta_1(t_1, u_1'(t_1))$$

$$\leq sp(t_1) - f(t_1, \alpha(t_1), \alpha'(t_1), \alpha''(t_1)) + u_1'(t_1) - \alpha'(t_1)$$

$$\leq sp(t_1) - f(t_1, \alpha(t_1), \alpha'(t_1), \alpha''(t_1)) < \alpha'''(t_1)$$

is achieved. Therefore $u_1'(t) > \alpha'(t)$, for $t \in [0, 1]$. In a similar way it can be proved that $u_1'(t) < \beta'(t)$, for every $t \in [0, 1]$ and so $u_1 \in \Omega$.

As the equations $Lu + F_\lambda u = 0$ and $Lu + N_\lambda u = 0$ are equivalent on $\Omega$ then

$$d(L + F_\lambda, \Omega_1) = d(L + F_\lambda, \Omega) = d(L + N_\lambda, \Omega) = \pm 1,$$

by (24) and the excision property of the degree. \hfill \Box

The main result is attained assuming that $f$ is bounded from below and it satisfies some adequate condition of monotonicity-type which requires different "speeds" of growth.

**Theorem 9.** Let $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function such that the assumptions of Theorem 5 are fulfilled. Suppose that there is $M > -r$ such that every solution $u$ of (E$_s$)–(3), with $s \leq s_1$, satisfies

$$u'(t) < M, \quad \forall t \in [0, 1].$$

(25)
and there exists \( m \in \mathbb{R} \) such that
\[
f(t, x, y, z) \geq mp(t),
\]
for every \((t, x, y, z) \in [0, 1] \times [-r, |M|] \times [-r, M] \times \mathbb{R}, \) with \( r \) given by (17). Then \( s_0, \) provided by Theorem 5, is finite and

1. if \( s < s_0, \) \((E_s)-(3)\) has no solution;
2. if \( s = s_0, \) \((E_s)-(3)\) has at least one solution.

Moreover, let \( M_1 := \max\{r, |M|\} \) and assume that there is \( \theta > 0 \) such that, for every \((t, x, y, z) \in [0, 1] \times [-M_1, M_1]^2 \times \mathbb{R} \) and \( 0 \leq \eta \leq 1, \)
\[
f(t, x + \eta\theta, y + \theta, z) \leq f(t, x, y, z).
\]

Then

3. for \( s \in [s_0, s_1], \) \((E_s)-(3)\) has at least two solutions.

**Proof.** Step 1. Every solution \( u(t) \) of \((E_s)-(3), \) for \( s \in [s_0, s_1], \) satisfies \(-r < u'(t) < M \) and \(-r < u(t) < |M|, \) with \( r \) given by (17) and \( t \in [0, 1]. \)

For first condition, by (25), it will be enough to show that \(-r < u'(t), \) for every \( t \in [0, 1] \) and for every solution \( u \) of \((E_s)-(3), \) with \( s \leq s_1. \)

Suppose, by contradiction, that there are \( s \in [s_0, s_1], \) a solution \( u \) of \((E_s)-(3)\) and \( t_2 \in [0, 1] \) such that
\[
u'(t_2) := \min_{t \in [0, 1]} u'(t) \leq -r.
\]

By (3), \( t_2 \in [0, 1], u''(t_2) = 0 \) and \( u'''(t_2) \geq 0. \) By (16),
\[
0 \leq u'''(t_2) = sp(t_2) - f(t_2, u(t_2), u'(t_2), u''(t_2)) \leq s_1 p(t_2) - f(t_2, u(t_2), -r, 0).
\]

If \( u(t_2) < -r, \) from (17) the following contradiction:
\[
0 \leq s_1 p(t_2) - f(t_2, u(t_2), -r, 0) \leq s_1 p(t_2) - f(t_2, -r, -r, 0) < 0
\]
is obtained. If \( u(t_2) \geq -r, \) from (15) and (17), the same contradiction is achieved. Then every solution \( u \) of \((E_s)-(3), \) with \( s_0 < s \leq s_1, \) verifies
\[
u'(t) > -r, \quad \forall t \in [0, 1].
\]

So, by (25), \(-r < u(t) < M, \) for every \( t \in [0, 1]. \) Integrating on \([0, t], \) we obtain
\[
-r \leq -rt < u(t) < Mt \leq |M|, \quad \forall t \in [0, 1].
\]

Step 2. The number \( s_0 \) is finite.

Suppose that \( s_0 = -\infty, \) that is, by Theorem 5, for every \( s \leq s_1 \) problem \((E_s)-(3)\) has at least a solution. Define \( p_1 := \min_{t \in [0, 1]} p(t) > 0 \) and take \( s \) sufficiently negative such that
\[
m - s > 0 \quad \text{and} \quad \frac{(m - s)p_1}{16} > M.
\]

If \( u(t) \) is a solution of \((E_s)-(3), \) then, by (26),
\[
u'''(t) = sp(t) - f(t, u(t), u'(t), u''(t)) \leq (s - m)p(t)
\]
and, by (3), there is \( t_3 \in [0, 1] \) such that \( u''(t_3) = 0. \) For \( t < t_3 \)
\[
u''(t) = \int_{t_3}^{t} u''(\xi) d\xi \geq \int_{t}^{t_3} (m - s)p(\xi) d\xi \geq (m - s)(t_3 - t)p_1.
\]
For $t \geq t_3$

$$u''(t) = \int_{t_3}^{t} u'''(\xi) d\xi \leq (s - m)(t - t_3)p_1.$$  

Choose $I = [0, \frac{1}{4}]$, or $I = [\frac{3}{4}, 1]$, such that $|t_3 - t| \geq \frac{1}{4}$, for every $t \in I$. If $I = [0, \frac{1}{4}]$, then

$$u''(t) \geq \frac{(m - s)p_1}{4}, \quad \forall t \in I,$$

and if $I = [\frac{3}{4}, 1]$, then

$$u''(t) \leq \frac{(s - m)p_1}{4}, \quad \forall t \in I.$$  

In the first case,

$$0 = \int_{0}^{\frac{1}{4}} u''(t) dt = \frac{1}{4} \int_{0}^{\frac{1}{4}} u''(t) dt + \int_{\frac{1}{4}}^{1} u''(t) dt \geq \frac{1}{4} \int_{0}^{\frac{1}{4}} (m - s)p_1 dt - u'(\frac{1}{4})$$

$$= \frac{1}{16} (m - s)p_1 - u'(\frac{1}{4}) > M - u'(\frac{1}{4}),$$

which is in contradiction with (25).

For $I = [\frac{3}{4}, 1]$ a similar contradiction is achieved. Therefore, $s_0$ is finite.

Step 3. For $s \in [s_0, s_1]$ $(E_s)$–(3) has at least two solutions.

As $s_0$ is finite, by Theorem 5, for $s_{-1} < s_0$, $(E_{s_{-1}})$–(3) has no solution. By Lemma 2 and Remark 1, we can consider $\rho_2 > 0$ large enough such that the estimate $\|u''\| < \rho_2$ holds for every solution $u$ of $(E_s)$–(3), with $s \in [s_{-1}, s_1]$.

Let $M_1 := \max \{r, |M|\}$ and define the set

$$\Omega_2 = \{x \in \text{dom } L: \|x'\| < M_1, \|x''\| < \rho_2\}.$$  

Then

$$d(L + N_{s_{-1}}, \Omega_2) = 0.$$  

By Step 1, if $u$ is a solution of $(E_s)$–(3), with $s \in [s_{-1}, s_1]$, then $u \notin \partial \Omega_2$. Defining the convex combination of $s_1$ and $s_{-1}$ as $H(\lambda) = (1 - \lambda)s_{-1} + \lambda s_1$ and considering the corresponding homotopic problems $(E_{H(\lambda)})$–(3), the degree $d(L + N_{H(\lambda)}, \Omega_2)$ is well defined for every $\lambda \in [0, 1]$ and for every $s \in [s_{-1}, s_1]$. Therefore, by (28) and the invariance of the degree

$$0 = d(L + N_{s_{-1}}, \Omega_2) = d(L + N_{s}, \Omega_2),\quad (29)$$

for $s \in [s_{-1}, s_1]$.

Let $\sigma \in [s_0, s_1] \subset [s_{-1}, s_1]$ and $u_\sigma(t)$ be a solution of $(E_\sigma)$–(3), which exists by Theorem 5. Take $\varepsilon > 0$ such that

$$|u'_\sigma(t) + \varepsilon| < M_1, \quad \forall t \in [0, 1].$$  

Then $\tilde{u}(t) := u_\sigma(t) + \varepsilon$ is a strict upper solution of $(E_\sigma)$–(3), with $\sigma < s \leq s_1$. In fact, by (27) with $\theta = \varepsilon$ and $\eta = t$, for such $\sigma$,

$$\tilde{u}''(t) = u''_\sigma(t) = \sigma p(t) - f(t, u_\sigma(t), u'_\sigma(t), u''_\sigma(t))$$

$$< sp(t) - f(t, u_\sigma(t), u'_\sigma(t), \tilde{u}''(t))$$

$$\leq sp(t) - f(t, u_\sigma(t) + \varepsilon t, u'_\sigma(t) + \varepsilon, \tilde{u}''(t))$$

$$= sp(t) - f(t, \tilde{u}(t), \tilde{u}'(t), \tilde{u}''(t)),$$

$$\tilde{u}(0) = 0, \quad \tilde{u}'(0) = \tilde{u}'(1) = \varepsilon > 0.$$
Moreover \( \alpha(t) := -r \) is a strict lower solution of \((E_s)-(3)\), for \( s \leq s_1 \). Indeed, by (17) and (15),
\[
\begin{align*}
\alpha''(t) &= 0 > s_1 p(t) - f(t, -r, -r, 0) \geq s p(t) - f(t, -rt, -r, 0), \\
\alpha(0) &= 0, \quad \alpha'(0) = \alpha'(1) = -r < 0.
\end{align*}
\]
By Step 1, \( -r < u'_s(t) \) for every \( t \in [0, 1] \) and therefore \( -r < u'_s'(t) + \varepsilon, \forall t \in [0, 1] \), that is, \( \alpha'(t) < u'(t) \). Integrating on \([0, 1]\)
\[
\alpha(t) \leq \alpha(t) - \alpha(0) < u(t) - \tilde{u}(0) = \tilde{u}(t),
\]
for every \( t \in [0, 1] \).

Then, by (30), Lemma 8 and Remark 2, there is \( \tilde{\rho}_2 > 0 \), independent of \( s \), such that for
\[
\Omega_\varepsilon = \{ x \in \text{dom} L: \alpha(t) < x(t) < \tilde{u}(t), \; \alpha'(t) < x'(t) < \tilde{u}'(t), \; \| x'' \| < \tilde{\rho}_2 \}
\]
the degree of \( L + N_s \) in \( \Omega_\varepsilon \) satisfies
\[
d(L + N_s, \Omega_\varepsilon) = \pm 1, \quad \text{for } s \in [\sigma, s_1].
\]
Taking \( \rho_2 \) in \( \Omega_2 \) large enough such that \( \Omega_\varepsilon \subset \Omega_2 \), by (29), (30) and the additivity of the degree, we obtain
\[
d(L + N_s, \Omega_2 - \Omega_\varepsilon) = \mp 1, \quad \text{for } s \in [\sigma, s_1].
\]
So, problem \((E_s)-(3)\) has at least two solutions \( u_1, u_2 \) such that \( u_1 \in \Omega_\varepsilon \) and \( u_2 \in \Omega_2 - \Omega_\varepsilon \), for \( s \in [s_0, s_1] \), since \( \sigma \) is arbitrary in \([s_0, s_1]\).

Step 4. For \( s = s_0 \), \((E_{s_0})-(3)\) has at least one solution.

Consider a sequence \( (s_m) \) with \( s_m \in [s_0, s_1] \) and \( \lim s_m = s_0 \). By Theorem 5, for each \( s_m \), \((E_{s_m})-(3)\) has a solution \( u_m \). Using the estimates of Step 1, it is clear that \( \| u_m \| < M_1 \), \( \| u'_m \| < M_1 \) independently of \( m \), and, by Remark 1, there is \( \tilde{\rho}_2 > 0 \) large enough such that \( \| u''_m \| < \tilde{\rho}_2 \), independently of \( m \). Then sequences \( (u_m) \) and \( (u'_m), m \in \mathbb{N}, \) are bounded in \( C([0, 1]) \). By the Arzelà–Ascoli theorem, we can take a subsequence of \( (u_m) \) that converges in \( C^2([0, 1]) \) to a solution \( u_0(t) \) of \((E_{s_0})-(3)\).

Hence, there is at least one solution for \( s = s_0 \). \( \Box \)

A variant of Theorem 9 can be obtained replacing \( f \) by \( -f \), \( x \) by \( -x \) and \( y \) by \( -y \).

**Theorem 10.** Consider \( f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R} \) a continuous function such that the assumptions of Theorem 6 are fulfilled. Suppose that there is \( M > -r \) such that every solution \( u \) of \((E_s)-(3)\), with \( s \geq s_1 \), satisfies
\[
u'(t) > M, \quad \forall t \in [0, 1],
\]
and there exists \( m \in \mathbb{R} \) such that
\[
f(t, x, y, z) \leq mp(t),
\]
for every \( (t, x, y, z) \in [0, 1] \times [-r, |M|] \times [-r, M] \times \mathbb{R} \). Then \( s_0 \) provided by Theorem 6 is finite and

1. if \( s > s_0 \), \((E_s)-(3)\) has no solution;
2. if \( s = s_0 \), \((E_s)-(3)\) has at least one solution.

Moreover, if condition (27) holds then

3. for \( s \in [s_1, s_0] \), \((E_s)-(3)\) has at least two solutions.

**Example.** Consider a particular case of problem (12)–(13) with \( n = m = 1, \; k = 4, \; b = d = B = C = 0, \; a, c > 0 \) and \( p(t) \equiv 1 \), that is
\[
(P) \quad \begin{cases}
\begin{align*}
u''(t) + |\nu''(t)|^\mu - 4(\nu'(t))^3 + (\nu(t))^3 &= s, \\
u(0) &= \nu'(0) = \nu'(1) = 0.
\end{align*}
\end{cases}
\]
with $\mu \in [0, 2]$. The function $f(t, x, y, z) = |z|^\mu - 4y^3 + x^3$ is continuous, verifies the Nagumo-type assumptions in $E$, given by (14), and monotonicity conditions (15) and (16). Consider $s_1$ and $r > 0$ large enough such that

$$0 < s_1 < f(t, x, -r, 0) = 4r^3 + x^3$$

holds for every $x \leq -r$. Therefore by Theorem 5 there is $s_0 < s_1$ such that $(P)$ has no solution for $s < s_0$ (if $s_0 = -\infty$, $(P)$ has a solution for every $s < s_1$) and for $s_0 < s < s_1$ problem $(P)$ has at least a solution.

For $r_*$ given by Lemma 2 define the set

$$E_1 = \{(t, x, y, z) \in [0, 1] \times \mathbb{R}^3: |x| \leq 1, |y| \leq 1, |z| \leq r_*\} \subset E.$$

Therefore, following the arguments of the proof of Theorem 4, for $f: E_1 \to \mathbb{R}$ every solution $u$ of $(P)$ verifies $|\mu'(t)| \leq 1$ in $[0, 1]$ and condition (26) holds with $m = -(5 + r_*^\mu)$. Moreover, for $0 \leq \eta \leq 1$ and $\theta \geq 5 + \sqrt{29}/2$, the inequality

$$f(t, x + \eta \theta, y + \theta, z) = (x + \eta \theta)^3 - 4(y + \theta)^3 + |z|^\mu \leq f(t, x, y, z)$$

is verified for $(t, x, y, z) \in [0, 1] \times [-1, 1]^2 \times \mathbb{R}$. So, by Theorem 9, $s_0$ is finite and for $s_0 < s \leq s_1$ problem $(P)$ has at least two solutions.

References