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Existence Result for Some Third Order Separated Boundary Value Problems

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Abstract

This paper provides conditions on a continuous function $f : [0,1] \times \mathbb{R}^3 \mapsto \mathbb{R}$ in order to obtain an existence and location result for the third order separated boundary value problem

$$\begin{cases} u'''(t) + f(t, u(t), u'(t), u''(t)) = 0, \\ u(0) = 0, \\ au'(0) - bu''(0) = A, \\ cu'(1) + du''(1) = B, \end{cases}$$

with $a, c, A, B \in \mathbb{R}$ and $b, d \geq 0$ such that $a^2 + b^2 > 0$ and $c^2 + d^2 > 0$. Nagumo conditions, lower and upper solutions, *a priori* estimates and Leray-Schauder degree play an important role in the arguments.

Key words: Third order separated boundary value problem, Nagumo-type conditions, lower and upper solutions, Leray-Schauder degree.

1 Introduction

This work presents an existence result for the third order separated boundary value problem

$$u'''(t) + f(t, u(t), u'(t), u''(t)) = 0 \tag{1}$$

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$$\begin{aligned} u(0) &= 0, \\ au'(0) - bu''(0) &= A, \\ cu'(1) + du''(1) &= B, \end{aligned} \tag{2}$$

where $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and $a, b, c, d, A, B \in \mathbb{R}$, $b, d \geq 0$ satisfy $a^2 + b^2 > 0$ and $c^2 + d^2 > 0$.

We apply for the above third order problem the lower and upper solutions method, following some arguments suggested by [4] and [3] to study some second order separated boundary value problems. This method also gives information about the location of the solutions obtained and of their derivatives. We refer [7] for an existence result concerning a separated boundary value problem for a n-th order equation, with $A = 0 = B$. There, the nonlinearity f depends on t, u and also on the derivatives of u but till the $(n-2)$-th order. Some results related to other third order boundary value problems can be found in [1], [2], [6] and in the references there contained.

Nagumo-type conditions, *a priori* estimates and Leray-Schauder degree play an important role in the arguments.

2 Preliminary Results

Given a subset $E \subseteq [0,1] \times \mathbb{R}^3$, a continuous function $g : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is said to satisfy Nagumo-type conditions in E if, for some $a > 0$, there is a continuous function $h : \mathbb{R}_0^+ \to [a, +\infty[$ such that

$$|g(t,x,y,z)| \leq h(|z|), \forall (t,x,y,z) \in E, \tag{3}$$

with

$$\int_0^{+\infty} \frac{s}{h(s)} ds = +\infty. \tag{4}$$

Lemma 1 *Let $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function that satisfies Nagumo-type conditions (3) and (4) in*

$$E = \left\{(t,x,y,z) \in [0,1] \times \mathbb{R}^3 : \Gamma_1(t) \leq x \leq \Gamma_2(t), \gamma_1(t) \leq y \leq \gamma_2(t)\right\},$$

where $\Gamma_1, \Gamma_2, \gamma_1, \gamma_2 : [0,1] \to \mathbb{R}$ *are continuous functions such that* $\Gamma_1(t) \leq \Gamma_2(t)$ *and* $\gamma_1(t) \leq \gamma_2(t)$, *for every* $t \in [0,1]$. *Then there exists* $r > 0$ *(depending only on* γ_1, γ_2 *and* h*) such that every solution* $u(t)$ *of equation (1), verifying for* $t \in [0,1]$

$$\Gamma_1(t) \leq u(t) \leq \Gamma_2(t),$$
$$\gamma_1(t) \leq u'(t) \leq \gamma_2(t),$$

satisfies $\|u''\|_\infty \leq r$.

Proof: Define the non-negative number

$$\eta = \max\{\gamma_2(1) - \gamma_1(0), \gamma_2(0) - \gamma_1(1)\}$$

and take $r > 0$ such that

$$\int_\eta^r \frac{s}{h(s)} ds \geq \max_{t\in[0,1]} \gamma_2(t) - \min_{t\in[0,1]} \gamma_1(t).$$

Let $\overline{u}(t)$ be a solution of (1) such that for $t \in [0,1]$

$$\Gamma_1(t) \leq \overline{u}(t) \leq \Gamma_2(t),$$
$$\gamma_1(t) \leq \overline{u}'(t) \leq \gamma_2(t).$$

Suppose, by contradiction, that $|\overline{u}''(t)| > \eta$, for every $t \in [0,1]$. If $\overline{u}''(t) > \eta$, $\forall t \in [0,1]$, we get the contradiction

$$\gamma_2(1) - \gamma_1(0) \geq \overline{u}'(1) - \overline{u}'(0) = \int_0^1 \overline{u}''(t)\, dt > \int_0^1 \eta\, dt \geq \gamma_2(1) - \gamma_1(0).$$

A similar contradiction can be obtained if we suppose that $\overline{u}''(t) < -\eta$, for every $t \in [0,1]$. So, there is $t \in [0,1]$ such that $|\overline{u}''(t)| \leq \eta$. If $|\overline{u}''(t)| \leq \eta$, for every $t \in [0,1]$, it is enough to take $r := \eta$ and the proof is finished. Suppose that $\overline{u}''(t) > \eta$, for some $t \in [0,1]$, and consider an interval $I = [t_0, t_1]$, or $I = [t_1, t_0]$, such that $\overline{u}''(t) \geq 0$, for every $t \in I$, $\overline{u}''(t_0) = \eta$ and $\overline{u}''(t) > \eta$, for every $t \in I \backslash \{t_0\}$. By a convenient change of variable, we have

$$\int_{\overline{u}''(t_0)}^{\overline{u}''(t_1)} \frac{s}{h(s)}\, ds = \int_{t_0}^{t_1} \frac{\overline{u}''(t)}{h(\overline{u}''(t))} \overline{u}'''(t)\, dt = \int_{t_0}^{t_1} \frac{-f(t, \overline{u}(t), \overline{u}'(t), \overline{u}''(t))}{h(\overline{u}''(t))} \overline{u}''(t)\, dt \leq$$
$$\leq \int_{t_0}^{t_1} \overline{u}''(t)\, dt = \overline{u}'(t_1) - \overline{u}'(t_0) \leq \max_{t\in[0,1]} \gamma_2(t) - \min_{t\in[0,1]} \gamma_1(t) \leq$$
$$\leq \int_\eta^r \frac{s}{h(s)}\, ds.$$

Then $\overline{u}''(t_1) \leq r$ and, since t_1 was taken arbitrarily as long as $|\overline{u}''(t)| > \eta$ in I, we conclude that for every $t \in [0,1]$ such that $\overline{u}''(t) > \eta$ we have $\overline{u}''(t) \leq r$.

Following similar steps it can be seen that $\overline{u}''(t) \geq -r$ for every $t \in [0,1]$ such that $\overline{u}''(t) < -\eta$. □

Let us define lower and upper solutions of (1)-(2):

Definition 1 *A function $\alpha(t) \in C^3([0,1])$ is a lower solution of problem (1)-(2) if*

$$\alpha'''(t) + f(t, \alpha(t), \alpha'(t), \alpha''(t)) \geq 0 \tag{5}$$

and

$$\begin{aligned} \alpha(0) &= 0, \\ a\alpha'(0) - b\alpha''(0) &\leq A, \\ c\alpha'(1) + d\alpha''(1) &\leq B. \end{aligned} \tag{6}$$

A function $\beta(t) \in C^3([0,1])$ is an upper solution of problem (1)-(2) if

$$\beta'''(t) + f(t, \beta(t), \beta'(t), \beta''(t)) \leq 0 \tag{7}$$

and

$$\begin{aligned} \beta(0) &= 0, \\ a\beta'(0) - b\beta''(0) &\geq A, \\ c\beta'(1) + d\beta''(1) &\geq B. \end{aligned} \tag{8}$$

3 Existence and Location Result

Theorem 1 *Let $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. Suppose that there exist lower and upper solutions of (1)-(2), $\alpha(t)$ and $\beta(t)$, respectively, such that, for $t \in [0,1]$,*

$$\alpha'(t) \leq \beta'(t) \tag{9}$$

and, for $(t, y, z) \in [0,1] \times \mathbb{R}^2$ *and* $\alpha(t) \le x \le \beta(t)$,

$$f(t, \alpha(t), y, z) \le f(t, x, y, z) \le f(t, \beta(t), y, z). \tag{10}$$

Suppose also that f *satisfies Nagumo-type conditions (3) and (4) in*

$$E_* = \left\{(t, x, y, z) \in [0,1] \times \mathbb{R}^3 : \alpha(t) \le x \le \beta(t), \alpha'(t) \le y \le \beta'(t)\right\}.$$

Then problem (1)-(2) has at least one solution $u(t) \in C^3([0,1])$ *satisfying*

$$\alpha(t) \le u(t) \le \beta(t) \quad \textit{and} \quad \alpha'(t) \le u'(t) \le \beta'(t), \quad \textit{for} \quad t \in [0,1].$$

We observe that an order relation between $\alpha(t)$ and $\beta(t)$ on $[0,1]$ is easily obtained from (9) by integration.

Proof: Given $w_1, w_2, w_3 \in \mathbb{R}$ such that $w_1 \le w_3$ define

$$\delta(w_1, w_2, w_3) = \begin{cases} w_3 & \text{if} \quad w_2 > w_3 \\ w_2 & \text{if} \quad w_1 \le w_2 \le w_3 \\ w_1 & \text{if} \quad w_2 < w_1. \end{cases}$$

For $\lambda \in [0,1]$ consider the auxiliary equation

$$\begin{aligned} &u'''(t) + \lambda f(t, \delta(\alpha(t), u(t), \beta(t)), \delta(\alpha'(t), u'(t), \beta'(t)), u''(t)) = \\ &= (1-\lambda)u'(t) + \lambda\left[u'(t) - \delta(\alpha'(t), u'(t), \beta'(t))\right] h(|u''(t)|) \end{aligned} \tag{11}$$

with the boundary conditions

$$\begin{aligned} u(0) &= 0, \\ u'(0) &= \lambda[A - a\delta(\alpha'(0), u'(0), \beta'(0)) + bu''(0) + \\ &\qquad + \delta(\alpha'(0), u'(0), \beta'(0))], \\ u'(1) &= \lambda[B - c\delta(\alpha'(1), u'(1), \beta'(1)) - du''(1) + \\ &\qquad + \delta(\alpha'(1), u'(1), \beta'(1))]. \end{aligned} \tag{12}$$

Take $r_1 > 0$ such that, for every $t \in [0,1]$,

$$-r_1 < \alpha'(t) \le \beta'(t) < r_1, \tag{13}$$

$$-f(t, \alpha(t), \alpha'(t), 0) - [r_1 + \alpha'(t)]h(0) < 0, \tag{14}$$

$$-f(t,\beta(t),\beta'(t),0)+[r_1-\beta'(t)]h(0)>0, \tag{15}$$

$$|A-a\beta'(0)+\beta'(0)|<r_1, |A-a\alpha'(0)+\alpha'(0)|<r_1, \tag{16}$$

$$|B-c\beta'(1)+\beta'(1)|<r_1, |B-c\alpha'(1)+\alpha'(1)|<r_1. \tag{17}$$

Step 1. *Every solution $u(t)$ of (11)-(12) satisfies in $[0,1]$*

$$|u(t)|<r_1 \qquad \text{and} \qquad |u'(t)|<r_1,$$

independently of λ.

Suppose by contradiction that the estimate $|u'(t)|<r_1$ is not true. Then there exists $t\in[0,1]$ such that $u'(t)\geq r_1$ or $u'(t)\leq -r_1$. Suppose that the first case holds. Define

$$\max_{t\in[0,1]} u'(t):=u'(t_0)(\geq r_1>0).$$

If $t_0\in]0,1[$, then $u''(t_0)=0$ and $u'''(t_0)\leq 0$. So, for $\lambda\in]0,1]$, by (10) and (15), we have the following contradiction

$$\begin{aligned}
0 \geq\ & u'''(t_0)= \\
&=-\lambda f(t_0,\delta(\alpha(t_0),u(t_0),\beta(t_0)),\delta(\alpha'(t_0),u'(t_0),\beta'(t_0)),u''(t_0))+ \\
&\quad +(1-\lambda)u'(t_0)+\lambda\left[u'(t_0)-\delta(\alpha'(t_0),u'(t_0),\beta'(t_0))\right]h(|u''(t_0)|) \\
&=-\lambda f(t_0,\delta(\alpha(t_0),u(t_0),\beta(t_0)),\beta'(t_0),0)+ \\
&\quad +(1-\lambda)u'(t_0)+\lambda\left[u'(t_0)-\beta'(t_0)\right]h(0) \\
&\geq -\lambda f(t_0,\beta(t_0),\beta'(t_0),0)+(1-\lambda)u'(t_0)+\lambda\left[r_1-\beta'(t_0)\right]h(0) \\
&\geq \lambda[-f(t_0,\beta(t_0),\beta'(t_0),0)+(r_1-\beta'(t_0))h(0)]>0
\end{aligned}$$

and, for $\lambda=0$,

$$0\geq u'''(t_0)=u'(t_0)\geq r_1>0.$$

If $t_0=0$, then

$$\max_{t\in[0,1]} u'(t):=u'(0)(\geq r_1)$$

and

$$u''(0^+)=u''(0)\leq 0.$$

So, the following contradiction is obtained

$$\begin{aligned}
r_1 \leq\ & u'(0)= \\
&=\lambda[A-a\delta(\alpha'(0),u'(0),\beta'(0))+bu''(0)+\delta(\alpha'(0),u'(0),\beta'(0))]= \\
&=\lambda[A-a\beta'(0)+bu''(0)+\beta'(0)]\leq \\
&\leq \lambda[A-a\beta'(0)+\beta'(0)]\leq |A-a\beta'(0)+\beta'(0)|<r_1.
\end{aligned}$$

If $t_0=1$, we have

$$\max_{t\in[0,1]} u'(t):=u'(1)(\geq r_1)$$

and

$$u''(1^-) = u''(1) \geq 0.$$

As above, we get a contradiction since, by (12) and (17),

$$\begin{aligned} r_1 \leq u'(1) &= \lambda[B - c\delta(\alpha'(1), u'(1), \beta'(1)) - du''(1) + \delta(\alpha'(1), u'(1), \beta'(1))] \\ &= \lambda[B - c\beta'(1) - du''(1) + \beta'(1)] \leq \lambda[B - c\beta'(1) + \beta'(1)] \\ &\leq |B - c\beta'(1) + \beta'(1)| < r_1. \end{aligned}$$

Then $u'(t) < r_1$, for every $t \in [0,1]$. In a similar way one can prove that $u'(t) > -r_1$, for $t \in [0,1]$. Moreover, since $u(0) = 0$, the estimate $|u(t)| < r_1$ is easily obtained by integration.

Step 2. *There is $r_2 > 0$ such that every solution $u(t)$ of (11)-(12) satisfies*

$$|u''(t)| < r_2, \forall t \in [0,1],$$

independently of $\lambda \in [0,1]$.

If $u(t)$ is a solution of (11)-(12) then

$$\begin{aligned} &u'''(t) + \lambda f(t, \delta(\alpha(t), u(t), \beta(t)), \delta(\alpha'(t), u'(t), \beta'(t)), u''(t)) - \\ &-(1-\lambda)u'(t) - \lambda\left[u'(t) - \delta(\alpha'(t), u'(t), \beta'(t))\right] h(|u''(t)|) = 0. \end{aligned}$$

Consider the set

$$E_{r_1} = \left\{(t,x,y,z) \in [0,1] \times \mathbb{R}^3 : -r_1 \leq x \leq r_1, -r_1 \leq y \leq r_1\right\}$$

and the function $F_\lambda : E_{r_1} \to \mathbb{R}$ defined by

$$\begin{aligned} F_\lambda(t,x,y,z) := \lambda f(t, \delta(\alpha(t), x, \beta(t)), \delta(\alpha'(t), y, \beta'(t)), z) - \\ -(1-\lambda)y - \lambda\left[y - \delta(\alpha'(t), y, \beta'(t))\right] h(|z|). \end{aligned}$$

We show that F_λ satisfies the Nagumo-type conditions (3) and (4) in E_{r_1}, independently of $\lambda \in [0,1]$. In fact, as f satisfies (3) in E_*, then

$$\begin{aligned} |F_\lambda(t,x,y,z)| &\leq |f(t, \delta(\alpha(t), x, \beta(t)), \delta(\alpha'(t), y, \beta'(t)), z| + \\ &\quad + |y| + |y - \delta(\alpha'(t), y, \beta'(t))|\, h(|z|) \\ &< h(|z|) + r_1 + (|y| + |\delta(\alpha'(t), y, \beta'(t))|) h(|z|) \\ &< r_1 + (1 + 2r_1) h(|z|). \end{aligned}$$

Define $\overline{h}(z) := r_1 + (1 + 2\, r_1)\, h(|z|)$. So, F_λ satisfies condition (3) in E_{r_1} with $h(z)$ replaced by $\overline{h}(z)$. And $\overline{h}(z)$ satisfies (4) since

$$\int_0^{+\infty} \frac{z}{\overline{h}(z)}\, dz = \int_0^{+\infty} \frac{z}{r_1 + (1+2r_1)h(z)}\, dz \geq \int_0^{+\infty} \frac{z}{h(z)(1+2r_1+\frac{r_1}{a})}\, dz \geq$$

$$\geq \frac{1}{1+2r_1+\frac{r_1}{a}} \int_0^{+\infty} \frac{z}{h(z)}\, dz = +\infty.$$

Therefore, taking Step 1 into account and applying Lemma 1 with

$$\Gamma_1(t) = \gamma_1(t) = -r_1 \qquad \text{and} \qquad \Gamma_2(t) = \gamma_2(t) = r_1,$$

there is $r_2 > 0$ such that $|u''(t)| < r_2$, for every $t \in [0,1]$. Since r_1 and h do not depend on λ, we observe that the estimate $|u''(t)| < r_2$ is also independent of λ.

Step 3. *Problem (11)-(12), for $\lambda = 1$, has at least a solution $u_1(t)$.*

Let us define the operators

$$\mathcal{L} : C^3([0,1]) \subset C^2([0,1]) \longmapsto C([0,1]) \times \mathbb{R}^3$$

by

$$\mathcal{L}u = (u''', u(0), u'(0), u'(1))$$

and

$$\mathcal{N}_\lambda : C^2([0,1]) \longmapsto C([0,1]) \times \mathbb{R}^3$$

by

$$\mathcal{N}_\lambda u = (-\lambda f(t, \delta(\alpha(t), u(t), \beta(t)), \delta(\alpha'(t), u'(t), \beta'(t)), u''(t)) +$$
$$+ (1-\lambda)u' + \lambda\,[u' - \delta(\alpha'(t), u'(t), \beta'(t)]\, h(|u''|), 0, A_\lambda, B_\lambda)$$

with

$$A_\lambda := \lambda[A - a\delta(\alpha'(0), u'(0), \beta'(0)) + bu''(0) + \delta(\alpha'(0), u'(0), \beta'(0))]$$

and

$$B_\lambda := \lambda[B - c\delta(\alpha'(1), u'(1), \beta'(1)) - du''(1) + \delta(\alpha'(1), u'(1), \beta'(1))]).$$

As $\mathcal{L}^{-1}$ is compact, we can define the completely continuous operator

$$\mathcal{T}_\lambda : \left(C^2([0,1]), \mathbb{R}\right) \longmapsto \left(C^2([0,1]), \mathbb{R}\right)$$

by

$$\mathcal{T}_\lambda(u) = \mathcal{L}^{-1}\mathcal{N}_\lambda(u).$$

Consider the set

$$\Omega = \{x \in C^2([0,1]) : \|x\|_\infty < r_1, \|x'\|_\infty < r_1, \|x''\|_\infty < r_2\}.$$

By Steps 1 and 2, the degree $d(\mathcal{T}_\lambda, \Omega)$ is well defined for every $\lambda \in [0,1]$ and, by homotopy invariance,

$$d(\mathcal{T}_0, \Omega) = d(\mathcal{T}_1, \Omega).$$

As the equation $x = \mathcal{T}_0(x)$ has only the trivial solution, by degree theory,

$$d(\mathcal{T}_0, \Omega) = \pm 1.$$

Hence, in particular, the equation $u = \mathcal{T}_1(u)$ has at least one solution. That is, the problem

$$\begin{aligned} u'''(t) + f(t, \delta(\alpha(t), u(t), \beta(t)), \delta(\alpha'(t), u'(t), \beta'(t)), u''(t)) = \\ = [u'(t) - \delta(\alpha'(t), u'(t), \beta'(t))]\, h(|u''(t)|), \end{aligned} \tag{18}$$

with

$$\begin{aligned} & u(0) = 0, \\ & u'(0) = A - a\delta(\alpha'(0), u'(0), \beta'(0)) + bu''(0) + \delta(\alpha'(0), u'(0), \beta'(0)), \\ & u'(1) = B - c\delta(\alpha'(1), u'(1), \beta'(1)) - du''(1) + \delta(\alpha'(1), u'(1), \beta'(1)) \end{aligned} \tag{19}$$

has at least a solution $u_1(t)$ in Ω.

Step 4. *The function $u_1(t)$ is a solution of (1)-(2).*

In fact, the above solution $u_1(t)$ of (18)-(19) will be a solution of the initial problem (1)-(2), too, since it satisfies in $[0,1]$

$$\alpha(t) \le u_1(t) \le \beta(t)$$

and

$$\alpha'(t) \le u_1'(t) \le \beta'(t).$$

Suppose, by contradiction, that there is $t \in [0,1]$ such that $u_1'(t) > \beta'(t)$ and define

$$\max_{t \in [0,1]} [u_1'(t) - \beta'(t)] := u_1'(t_2) - \beta'(t_2) > 0.$$

If $t_2 \in]0,1[$ then

$$u_1''(t_2) = \beta''(t_2) \quad \text{and} \quad u_1'''(t_2) \le \beta'''(t_2).$$

However, this is not possible because, by (10),

$$\begin{aligned}
0 &\geq u_1'''(t_2) - \beta'''(t_2) \\
&\geq -f(t_2, \delta(\alpha(t_2), u_1(t_2), \beta(t_2)), \delta(\alpha'(t_2), u_1'(t_2), \beta'(t_2)), u_1''(t_2)) + \\
&\quad + [u_1'(t_2) - \delta(\alpha'(t_2), u_1'(t_2), \beta'(t_2))]\, h(|u_1''(t_2)|) + \\
&\quad + f(t_2, \beta(t_2), \beta'(t_2), \beta''(t_2)) \\
&= -f(t_2, \delta(\alpha(t_2), u_1(t_2), \beta(t_2)), \beta'(t_2), \beta''(t_2)) + \\
&\quad + [u_1'(t_2) - \beta'(t_2)]\, h(|u_1''(t_2)|) + f(t_2, \beta(t_2), \beta'(t_2), \beta''(t_2)) \\
&> -f(t_2, \beta(t_2), \beta'(t_2), \beta''(t_2)) + f(t_2, \beta(t_2), \beta'(t_2), \beta''(t_2)) \\
&= 0,
\end{aligned}$$

which is a contradiction.

If $t_2 = 0$ we have

$$\max_{t \in [0,1]} [u_1'(t) - \beta'(t)] := u_1'(0) - \beta'(0) > 0$$

and

$$u''(0^+) - \beta''(0^+) = u''(0) - \beta''(0) \leq 0.$$

Then, by (8), the following contradiction is obtained:

$$\begin{aligned}
\beta'(0) < u_1'(0) &= A - a\delta(\alpha'(0), u'(0), \beta'(0)) + bu_1''(0) + \delta(\alpha'(0), u'(0), \beta'(0)) \\
&= A - a\beta'(0) + bu_1''(0) + \beta'(0) \leq -b\beta''(0) + bu_1''(0) + \beta'(0) \leq \beta'(0).
\end{aligned}$$

Then $t_2 \neq 0$ and, with similar arguments, we deduce that $t_2 \neq 1$. So

$$u_1'(t) \leq \beta'(t), \; \forall t \in [0,1].$$

Applying an analogous technique, it can be proved that $\alpha'(t) \leq u_1'(t)$, for every $t \in [0,1]$.

Integrating the inequalities $\alpha'(t) \leq u_1'(t) \leq \beta'(t)$ on $[0,t]$, we obtain

$$\alpha(t) - \alpha(0) \leq u_1(t) - u_1(0) \leq \beta(t) - \beta(0),$$

that is,

$$\alpha(t) \leq u_1(t) \leq \beta(t), \; \forall t \in [0,1].$$

So $u_1(t)$ is a solution of (1)-(2). □

Example: Consider the separated boundary value problem

$$\begin{cases} u'''(t) + \arctan(u(t)) - \arctan(u'(t))\sqrt[3]{(u''(t))^2+1} = 0, \\ u(0) = 0, \\ au'(0) - bu''(0) = A, \\ cu'(1) + du''(1) = B, \end{cases} \tag{20}$$

with $a, b, c, d \geq 0$ such that $a^2 + b^2 > 0$ and $c^2 + d^2 > 0$.

The function $f(t, x, y, z) = \arctan(x) - \arctan(y)\ \sqrt[3]{z^2+1}$ is continuous.

If $A, B \in \mathbb{R}$ are such that $|A| \leq a$ and $|B| \leq c$, then the functions

$$\alpha, \beta : [0, 1] \to \mathbb{R}$$
$$\alpha(t) = -t \qquad \text{and} \qquad \beta(t) = t$$

are, respectively, lower and upper solutions for (20). Moreover, the function f satisfies Nagumo-type conditions (3) and (4) in

$$E = \left\{(t, x, y, z) \in [0, 1] \times \mathbb{R}^3 : -t \leq x \leq t, -1 \leq y \leq 1\right\}.$$

Conditions (9) and (10) are also satisfied. Therefore, by Theorem 1, there is at least a solution $u(t)$ for (20) such that, for every $t \in [0, 1]$,

$$-t \leq u(t) \leq t,$$

and

$$-1 \leq u'(t) \leq 1.$$

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