Hausdorff First Countable, Countably Compact Space is $\omega$-Bounded

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Abstract. In this paper we obtain an answer to the question in Problem 288, the book Open Problems in Topology by Jan van Mill and George M. Reed, p. 131. Namely, we prove that a Hausdorff first countable, countably compact topological space is $\omega$-bounded. We also point out errors occurring in the literature concerning the Ostaszewski spaces.

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1. Introduction

We prove that any Hausdorff ($C1$) countably compact space is in fact $\omega$-bounded. In the Preliminaries, (paragraph 2), we state the definitions concerning Problem 288 in [3] and properties that we use to obtain the main result (Corollary 1) in paragraph 3. This work is on Topology and assumes the Axiomatics of Set Theory as in [1, 2]; it does not concern the different, although related, difficult subject of Logic and the Theory of Sets, namely models in ZFC. Still in paragraph 3, we prove that it is wrong to assume that the $\gamma N$ space in [3, p. 133] is both Hausdorff and countably compact and that [5, p. 652] contains a contradiction; hence those matters do not really contradict Corollary 1. Also, we show
that separation is essential in Corollary 1.

2. Preliminaries

Recall that a topological space \((X, T)\) is first countable or a \((C1)\) space if each point has a countable base of neighborhoods. \((X, T)\) is called a \((T1)\) space if for every pair of distinct points \(x, y \in X\) there exist open sets \(W_x, W_y \subset X\) such that \(x \in W_x\) \((y \in W_y)\) whereas \(y \notin W_x\) \((x \notin W_y)\) and, it is a Hausdorff space or a \((T2)\) space if two any different points have disjoint neighborhoods. Clearly any \((T2)\) space is a \((T1)\) space.

**Definition 1.** (Following [1]) We say that a Hausdorff space \((X, T)\) is countably compact if any countable open cover of \(X\) has a finite subcover.

**Remark 1.** According to [1, Chap. XI, Sec. 3, p. 233] if \((X, T)\) is metrizable and countably compact, then \((X, T)\) is compact.

**Definition 2.** (Following [3, p. 131]) A topological space \((X, T)\) is said to be \(\omega\)-bounded if each countable subset of \(X\) has compact closure.

Following [2, p. 331] the base \(B\) for the topological space \((X, T)\) is called regular if for every point \(x \in X\) and any neighborhood \(U\) of \(x\) there is a neighborhood \(V \subset U\) of the point \(x\) such that the set of all members of \(B\) that meet both \(V\) and \(X \setminus U\) is finite. We then have the Arkhangel’skii metrization theorem

**Theorem 1.** A topological space is metrizable if and only if it is a \((T1)\) space and has a regular base.

**Proof.** See [2, 5.4.6., p. 332].

Recall that a partially ordered set \((M, \leq)\) is well-ordered if for each nonempty set \(\mathcal{B} \subset M\) there is some \(b_0 \in \mathcal{B}\) such that \(b_0 \leq b\) for each \(b \in \mathcal{B}\).

**Definition 3.** (Following [1]) An ordinal number is a set \(\alpha\) with the properties that, for each \(x, y \in \alpha\) such that \(x \neq y\) it holds that either \(x \in y\) or \(y \in x\) and, \((x \in y) \land (y \in \alpha) \Rightarrow x \in \alpha\).

Following [1] we say that a bijection \(f : (M, \leq) \rightarrow (\mathfrak{N}, \leq)\) between two well-ordered sets is an isomorphism if it holds that \(a \leq b\) implies \(f(a) \leq f(b)\). For each \(a \in \mathfrak{M}\), the set \(\mathfrak{M}(a) = \{x \in \mathfrak{M} : (x \leq a) \land (x \neq a)\}\) is the initial interval determined by \(a\). Also if \(\alpha\) is an ordinal number, the initial interval \(\mathfrak{C}(\alpha) = \alpha,\) where \(\mathfrak{C}\) is the well-ordered class of all ordinal numbers, well-ordered putting \(\alpha \leq \beta\) if and only if \(\alpha \subset \beta\). Each well-ordered set is isomorphic to a suitable \(\mathfrak{C}(\alpha)\). (See [1, pp. 36, 42, 43] and Theorem 6.4, at the same page). If we say that two sets \(A, B\) are equipotent meaning that there is a bijective map on \(A\) onto \(B\), it follows by Zermelo’s theorem [1, Th. 2.1 (3), p. 32] that in the class
of all equipotent sets to \( A \), there exists an ordinal number and, also the smallest such ordinal number, which is called the cardinal number of \( A \) ([1, p. 46]).

The first ordinal number is the empty set \( \emptyset \). For each ordinal number \( \alpha \), its successor is \( \alpha \cup \{\alpha\} = \alpha + 1 \). We denote \( \emptyset = 0 \), \( \{\emptyset\} = 1 \), \( \{\emptyset\} \cup \{\emptyset\} = 2 \), ... These are the finite ordinal numbers, which can be viewed as the natural numbers. The first infinite ordinal number is \( \omega \), which is the cardinal number of the set of all natural numbers. \( \omega \) is a limit ordinal number, i.e., it is not a successor of an ordinal number. We say that \( \omega \) is an infinite countable ordinal number and we denote by \( \omega_1 \) the first uncountable ordinal number; it holds that \( \omega_1 \leq c \) where ordinal \( c \) is the cardinal number of the set of all real numbers when viewed as an ordinal number. We shall also consider the ordinal number \( \omega_2 \), namely, we denote by \( \omega_2 \) the smallest cardinal number viewed as an ordinal number, that is greater than \( \omega_1 \).

Remark 2. We say that a set \( A \) has cardinality no greater than the continuum if there exists an injective map from \( A \) to a set which has the cardinality of the continuum \( c \). In the following we consider \( c \) viewed as an ordinal number: the continuum hypothesis, which is the assumption that \( \omega_1 = c \), is not required in what follows.

Definition 4. The ordinal space \([0, c]\) is the set of all ordinal numbers \( \alpha < c \) (we denote by \( \alpha < \beta \) meaning that \( \alpha \subset \beta \), \( \alpha \neq \beta \), \( \alpha, \beta \in \mathbb{C} \) as above), equipped with the topology that is generated by the sets of the form \( \{x : x \succ \alpha\} \) and \( \{x : x \prec \beta\} \).

Remark 3. The set of all limit ordinal numbers in \([0, c]\) has the cardinality of the continuum, in the sense of the preceding remark. Hence also the set of all limit ordinal numbers \( \alpha \) in \([0, c]\) such that \( \omega \prec \alpha \) has the cardinality of the continuum.

Remark 4. It holds that \( \{\alpha + 1\} = |\alpha, (\alpha + 1) + 1| \) is an open subset of \([0, c]\) for each \( \alpha \in [0, c] \).

Remark 5. In the sense of the above remarks, if the set \( A \) has cardinality no greater than the continuum, then there exists an injection from \( A \) to \([1, \omega] \cup \mathcal{I} \), where \( \mathcal{I} \) is a subset of \( |\omega, c| \) constituted by different limit ordinal numbers.

3. The Results

In the following, we consider a Hausdorff first countable, countably compact topological space \((X, T_X)\).

Lemma 1. Let \( C = \{x_n : n = 1, 2, \ldots\} \subset (X, T_X) \) be countably infinite. Then each point \( x \in \overline{C} \) is the limit \( x = \lim_{k} x_{n(k)} \) of a sequence \( (x_{n(k)}) \) in \( C \).

Proof. In fact, if \( x \) is the limit of a net \((\alpha)\) in \( C \) it follows that, for \( \{V_k : k = 1, 2, \ldots\} \) being a countable base of neighborhoods of \( x \) such that \( V_k \supset V_{k+1} \) for
Corollary 1. If $\kappa$ each $172$ assigning a fixed $\map x V$ point, holds that $\Pr$. Consider $\omega$ topological space then it is metrizable for a metric $d$. It holds that $\overline{C} = \Pr_2(E)$ by Lemma 1 and the definition of the set $S$, where the map $\Pr_2 : E \to X, \Pr_2(u, v) = v$ is a homeomorphism. In fact injectivity holds, because we assume that the $x_n$ are all different in Lemma 1; also the assignment $[n(k)] \to p = \lim_k x_n(k)$ is a bijection on the set $S$ of the $[n(k)]$ to the set of limits $\lim_k x_n(k)$, for $v = \lim_k x_j(k) \neq \lim_k x_n(k) = p$ implies that $S(v) \cap S(p) = \emptyset$ in the above notation, hence $[j(k)] \neq [n(k)], \alpha [n(k)] \neq \alpha [j(k)]$. Therefore the induced topology on $\overline{C}$ by $T_X$ is the topology for the metric $d_2$ defined through $d_2(v, v') = ^\to$$\\Pr$
d((u, v), (u', v')) if and only if \( pr_2(u, v) = v, pr_2(u', v') = v' (v, v' \in \overline{T}) \). The fact that the closure \( \overline{T} \) is compact follows from being countably compact ([1, Theorem 3.6 (2), p. 230]) and by Remark 1 and the corollary is proved.

\[ \text{Remark 6.} \quad \text{Any space whose topology is strictly finer than the Hausdorff first countable topology fails to be countably compact.} \]

\[ \text{Proof.} \quad \text{Let} \ (X, T) \text{ be first countable Hausdorff and let} \ \sigma \text{ be a topology on} \ X \text{ strictly finer than} \ T. \text{ This implies that there is a set} \ A \text{ whose}\sigma\text{-closure} \ A^\sigma \text{ is a proper subset of its} \ T \text{-closure} \ A^T. \text{ Let} \ p \in A^T \setminus A^\sigma. \text{ By first countability, there is a} \ T \text{-sequence in} \ A \text{ converging to} \ p. \text{ The range of the sequence has} \ p \text{ as its only} \ T \text{-accumulation point. Since} \ p \text{ is not in the} \ \sigma \text{-closure of the range, the range is an infinite closed discrete subspace of} \ (X, \sigma) \text{ and the remark follows.} \]

\[ \text{Remark 7.} \quad \text{The space} \ \gamma N \text{ defined at p. 133 and characterized in Example 2.2. of [3] is not first countable Hausdorff.} \]

\[ \text{Proof.} \quad \text{Just before Example 2.2 the author states that he considers a definition of} \ N \text{ that makes it disjoint form} \ \omega_1, \text{ so that he considers the Franklin-Rojagopalan space} \ \gamma N \text{ in such a way that, he identifies} \ \gamma N \setminus N \text{ they with} \ \omega_1. \text{ In Example 2.2, the author states namely "Let} \ \{A_\alpha : \alpha \in \omega_1\} \text{ be a} \ C^* \text{-ascending sequence of infinite subsets of} \ N. \text{ (An easy "diagonal" argument allows one to construct such a sequence in} \ ZFC. \text{ Set} \ A_{-1} = \phi. \text{ On the set} \ N \cup \omega_1 \text{ we impose the topology} \ T \text{ which has the sets of the form} \ \{n\} \ (n \in N) \text{ and} \ U_n(\beta, \alpha) \ (n \in N, \beta \in \omega_1 \cup \{-1\}, \alpha \in \omega_1) \text{ as a base, where} \ (\beta, \alpha) \text{ means} \ \{\gamma \in \omega_1 : \beta < \gamma \leq \alpha\} \text{ and} \ U_n(\beta, \alpha) = (\beta, \alpha) \cup (A_\alpha \setminus A_\beta) \setminus \{1, ..., n\} \text{ where} \ \{1, ..., n\} \subset N \ldots \text{ Thus this gives a} \ \gamma N \text{ "}. \text{ Here, the symbol} \ C^* \text{ stands for} \ B \subset^* A \text{ meaning that} \ B \setminus A \text{ is finite and} \ A \setminus B \text{ is infinite, where} \ A, B \text{ are subsets of} \ N \text{ (line 7). Now we have the following: clearly that in the notation as above, the sets} \ W_n(\beta, \alpha) = (\beta, \alpha) \setminus \{1, ..., n\} \text{ together with the sets} \ \{n\} \text{ constitute a base for a topology} \ \sigma \text{ that is strictly finer than the topology} \ T \text{ defined in Example 2.2 as above. It holds that,} \ \sigma \text{ is a countably compact topology ([1, Ex. 1, Chap. XI, Sec. 3, pp. 228, 229]) hence, by the preceding Remark,} \ T \text{ cannot be first countable Hausdorff and the remark follows.} \]

The author goes on in Example 2.2, showing that each \( \gamma N \) space can be obtained through a \( C^* \)-ascending sequence of infinite sets as he starts explaining in Example 2.2. Now at p. 134, Observation 2.3, the author asserts that namely,” \( \gamma N \) is countably compact if and only if, no infinite subset of \( N \) is almost disjoint from all the \( A_\alpha \). (Two sets are said to be almost disjoint if their intersection is finite). The existence of a \( C^* \)-ascending sequence \( \{A_\alpha : \alpha \preceq \omega_1\} \) of subsets of \( N \), such that no infinite subset of \( N \) is almost disjoint from all \( A_\alpha \), is equivalent to \( t = \omega_1 \). The cardinal number \( t \) above is defined at lines 8 and 9 p. 133 ([3]). Line 21 p. 135 in [3] asserts that it is not known if a Ostaszewski-van Dowen space exists in \( ZFC \). It follows from the above that it leads to a contradiction the method for the proof of 3.3. Construction, pp. 135, 136 using the extra hypothesis to \( ZFC \).
Concerning p. 652 in [5], we now prove that there is a flaw in the proof. The author considers spaces $X_\alpha = \omega \cup \{p_\beta : \beta < \alpha\}$ where, $\alpha \in \omega_1$, $X_{\omega_1} = \omega \cup \{X_\alpha : \alpha < \omega_1\}$. He then defines a topology $T$ on $X_{\omega_1}$ such that $X_{\omega_1}$ has $\omega$ as a dense subspace and, according to the proof, $\omega$ is dense in each space $X_\alpha$.

At line 21, we can read that the $X_\alpha$ are open in $X_{\omega_1}$; further, each $X_\alpha$ is metrizable (line 9) hence it is a Hausdorff space. If $\beta < \alpha$ then $X_\beta$ is a subspace of $X_\alpha$ (line 7) hence the topology of $X_\beta$ is the induced topology on $X_\beta$ by $X_\alpha$. $X_\alpha$ being locally compact (line 6) we may take the point $p_\alpha$ as \(\infty\) concerning the Alexandroff one point compactification $\hat{X}_\alpha$ so that $\hat{X}_\alpha = X_\alpha \cup \{p_\alpha\} = X_{\alpha+1}$ as sets. Clearly the identity injection $X_{\alpha+1} \rightarrow \hat{X}_\alpha$ is continuous (since $X_{\alpha+1}$ is separated Hausdorff). We prove that the point $p_{\alpha+2}$ is not the limit, in $X_{\omega_1}$, of a sequence in $\omega$; hence, $\omega$ is not dense in $X_{\omega_1}$, contradicting the conclusion at lines 20, 21. Let the $n(k)$ be in $\omega$, $n(k) \rightarrow p$. We have that $n(k) \rightarrow p$ in some space $X_\gamma$, $\alpha + 2 < \gamma < \omega_1$. Also, a subsequence $n(k(j)) \rightarrow q$ in $\hat{X}_\alpha$ hence $q \in X_{\alpha+1}$. It follow that $n(k(j)) \rightarrow p$ in the topology of $\hat{X}_\gamma$ and, $n(k(j)) \rightarrow q$ in the topology of $\hat{X}_\alpha$, hence $n(k(j)) \rightarrow q$ also in the topology of $\hat{X}_\gamma$; therefore $p = q \in X_{\alpha+1}$ which is a contradiction as we wished to point out. We also obtain the contradiction that $\omega$ is not dense in $X_\gamma$.

We now obtain an example of a (non separated) first countable, countably compact space that is not $\omega$-bounded.

**Remark 8.** If $\alpha \in [0, \omega_1[$ and $\alpha$ has an immediate predecessor $\alpha^-$ (that is, $\alpha = \alpha^- + 1$, $\alpha^-$ an ordinal number), we have that $|\alpha^-, \alpha + 1| = \{\alpha\}$ is open in $[0, \omega_1[$. If $\alpha$ has no immediate predecessor, then the set $\{\lambda, \alpha + 1 : \lambda \leq \alpha\}$ is a countable base of neighborhoods of $\alpha$. Hence $[0, \omega_1[$ is a first countable space.

**Remark 9.** (Following [1]) The ordinal space $[0, \omega_1[$ is countably compact and it is not compact.

**Proof.** See [1, Ex. 1, Chap. XI, Sec. 3, p. 228, 229].

Keeping the notations as in the Preliminaries, we consider the set of ordinal numbers $E = [0, \omega_2]\setminus\{\omega_1\}$.

**Example 1.** $(E, T^*)$ will stand for a set $E$ equipped with a topology $T^*$ for which a set $A \subseteq E$ is open if and only if both conditions hold: (1) if $\alpha \in A \cap [0, \omega_1[$ then $A \cap [0, \omega_1[$ contains an open subset of the ordinal space $[0, \omega_1[$ containing $\alpha$; (2) if $\gamma \in A \cap [\omega_1, \omega_2[$ then $A \supseteq [0, \omega_1, \omega_2[$. Then we have

**Claim 1.** The class $T^*$ is a non separated topology on $E$ such that, the induced topology on $[0, \omega_1[$ coincides with the topology of the ordinal space $[0, \omega_1[$.

**Proof.** This follows immediately.

**Claim 2.** The topological space $(E, T^*)$ is first countable, countably compact and it is not $\omega$-bounded.
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*Proof.* If $\gamma \in [\omega_1, \omega_2]$ then the class $\{[0,\omega]\cup[\omega_1,\gamma]\}$ is a base of neighborhoods of $\gamma$, hence the space is (C1) by Remark 8 and Claim 1. Using Remark 9 and E (d) in [4, p. 162] we conclude that $(E, T^\ast)$ is countably compact if we prove that each sequence $(\gamma_n)$ in $[\omega_1, \omega_2]$ has a convergent subsequence. Since each infinite increasing sequence in $[\omega_1, \omega_2]$ has an upper bound in $[\omega_1, \omega_2]$, we may obtain an increasing subsequence $(\gamma_n(k))$ of $(\gamma_n)$ (both these facts follow easily from the fact that $[\omega_1, \omega_2]$ is well ordered). Now $\gamma = \sup\{\gamma_n(k) : k = 1, 2, \ldots\}$ is the limit of $(\gamma_n(k))$ in $(E, T^\ast)$ according to the definition of the topology (an easy cardinality argument shows that $\gamma < \omega_2$) hence $(E, T^\ast)$ is countably compact. It remains to prove that the space is not ω-bounded. In fact, each point $\gamma \in [\omega_1, \omega_2]$ is in the closure of the countable set $[0, \omega]$; hence $[0, \omega] = E$. We have that $E$ is not compact (clearly the net $(\beta_\alpha)$, where $\beta_\alpha = \alpha \in [0, \omega_2]\backslash\{\omega_1\}$ has no convergent subnet) and the claim follows.

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**References**